

Quantizations of the extended affine Lie algebra $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}^*$

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Abstract. The extended affine Lie algebra $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$ is quantized from three different points of view in this paper, which produces three noncommutative and noncocommutative Hopf algebra structures, and yield other three quantizations by an isomorphism of $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$ correspondingly. Moreover, two of these quantizations can be restricted to the extended affine Lie algebra $\mathfrak{sl}_2(\mathbb{C}_q)$.

Key words: Quantizations, Lie bialgebras, Drinfel'd twists, the extended affine Lie algebra $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$

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1. Introduction

During the investigation of quantum groups, V. Drinfel'd introduced the notion of Lie bialgebras [8] in 1983. Quantization of Lie algebras and bialgebras is an important way to produce new quantum groups. Quantizations by twists act basically for constructing new quantized enveloping algebras. A universal and functional quantization of Lie bialgebras was developed in [11, 12] employing the Tannaka-Krein approach, from which a quantization of any finite dimensional Lie bialgebra defined over a field of characteristic zero (see [11]) was constructed. Although a general method for twisting both the product and coproduct of a bialgebra does not appear, it is possible to twist the corresponding coproduct in such a way that it remains compatible with its original multiplication, unit, and counit (see [18]). In this paper, we shall concentrate on the quantization being assort to the so-called Drinfel'd twist of the extended affine Lie algebra (EALA) $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$, whose Lie bialgebra structures were determined in [25]. The EALA $\mathfrak{sl}_2(\mathbb{C}_q)$ was first introduced in [19] in the sense of quasi-simple Lie algebras and systematically investigated in [2]. Since then, the representation and structure theory of such Lie algebras have been attentively studied (see [1, 3–6, 14–16, 22, 23] and the references therein).

We now introduce the Lie algebra considered in this paper. Denote \mathbb{Z} , \mathbb{Z}^* , \mathbb{C} the sets of all integers, nonzero integers, complex numbers respectively. Let $\mathbf{0} = (0, 0)$, $\mathbf{Z} = \mathbb{Z} \times \mathbb{Z}$, $\mathbf{Z}^* = \mathbb{Z}^* \times \mathbb{Z}^*$. For any $\mathbf{m} = (m_1, m_2) \in \mathbf{Z}$, $\mathbf{k} = (k_1, k_2) \in \mathbf{Z}^*$, introduce the following elements of $\mathfrak{L} = \mathfrak{sl}_2(\mathbb{C}_q)$:

$$\begin{aligned} e_{\mathbf{m}} &= E_{12}x^{m_1}y^{m_2}, & f_{\mathbf{m}} &= E_{21}x^{m_1}y^{m_2}, \\ d &= E_{11} - E_{22}, & g_{\mathbf{k}} &= E_{11}x^{k_1}y^{k_2}, & h_{\mathbf{k}} &= E_{22}x^{k_1}y^{k_2}, \end{aligned}$$

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which form a basis of \mathfrak{L} with the following relations:

$$\begin{aligned}
[e_{\mathbf{m}}, e_{\mathbf{m}'}] &= [f_{\mathbf{m}}, f_{\mathbf{m}'}] = [d, h_{\mathbf{k}}] = [d, g_{\mathbf{k}}] = [g_{\mathbf{k}}, h_{\mathbf{k}'}] = 0, \\
[g_{\mathbf{k}}, e_{\mathbf{m}}] &= q^{k_2 m_1} e_{\mathbf{k}+\mathbf{m}}, \quad [h_{\mathbf{k}}, e_{\mathbf{m}}] = -q^{k_1 m_2} e_{\mathbf{k}+\mathbf{m}}, \quad [d, e_{\mathbf{m}}] = 2e_{\mathbf{m}}, \\
[h_{\mathbf{k}}, f_{\mathbf{m}}] &= q^{k_2 m_1} f_{\mathbf{k}+\mathbf{m}}, \quad [g_{\mathbf{k}}, f_{\mathbf{m}}] = -q^{k_1 m_2} f_{\mathbf{k}+\mathbf{m}}, \quad [d, f_{\mathbf{m}}] = -2f_{\mathbf{m}}, \\
[e_{\mathbf{m}}, f_{\mathbf{m}'}] &= \begin{cases} q^{m_2 m'_1} g_{\mathbf{m}+\mathbf{m}'} - q^{m'_2 m_1} h_{\mathbf{m}+\mathbf{m}'} & \text{if } \mathbf{m} + \mathbf{m}' \neq \mathbf{0}, \\ q^{m_2 m'_1} d & \text{if } \mathbf{m} + \mathbf{m}' = \mathbf{0}, \end{cases} \\
[g_{\mathbf{k}}, g_{\mathbf{k}'}] &= \begin{cases} (q^{k_2 k'_1} - q^{k'_2 k_1}) g_{\mathbf{k}+\mathbf{k}'} & \text{if } \mathbf{k} + \mathbf{k}' \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{k} + \mathbf{k}' = \mathbf{0}, \end{cases} \\
[h_{\mathbf{k}}, h_{\mathbf{k}'}] &= \begin{cases} (q^{k_2 k'_1} - q^{k'_2 k_1}) h_{\mathbf{k}+\mathbf{k}'} & \text{if } \mathbf{k} + \mathbf{k}' \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{k} + \mathbf{k}' = \mathbf{0}. \end{cases}
\end{aligned}$$

Then $\mathfrak{L} = \oplus_{\mathbf{m}} \mathfrak{L}_{\mathbf{m}}$, where $\mathfrak{L}_{\mathbf{0}} = \mathbb{C}e_{\mathbf{0}} \oplus \mathbb{C}f_{\mathbf{0}} \oplus \mathbb{C}d$, $\mathfrak{L}_{\mathbf{k}} = \mathbb{C}e_{\mathbf{k}} \oplus \mathbb{C}f_{\mathbf{k}} \oplus \mathbb{C}g_{\mathbf{k}} \oplus \mathbb{C}h_{\mathbf{k}}$. Introduce two degree derivations d_1 and d_2 on \mathfrak{L} :

$$[d_1, L] = m_1 L, \quad [d_2, L] = m_2 L \quad \text{for } L \in \mathfrak{L}_{\mathbf{m}} \quad \text{and} \quad [d_1, d_2] = [d, d_1] = [d, d_2] = 0.$$

Then we arrive at the EALA $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)} = \mathfrak{L} \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$ considered in this paper and denoted by $\widetilde{\mathfrak{L}}$ for convenience. Also $\widetilde{\mathfrak{L}}$ is \mathbf{Z} -graded: $\widetilde{\mathfrak{L}} = \oplus_{\mathbf{m} \in \mathbf{Z}} \widetilde{\mathfrak{L}}_{\mathbf{m}}$ with $\widetilde{\mathfrak{L}}_{\mathbf{k}} = \mathfrak{L}_{\mathbf{k}}$ for $\mathbf{k} \in \mathbf{Z}^*$ and $\widetilde{\mathfrak{L}}_{\mathbf{0}} = \mathfrak{L}_{\mathbf{0}} \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$. Denote the universal enveloping algebra of the Lie algebra $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$ by $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})$.

For $\mathbf{m} = (m_1, m_2)$, $\mathbf{n} = (n_1, n_2)$, $i \in \mathbb{Z}$, introduce the following notations that will be referred to in the main theorem:

$$\begin{aligned}
\gamma_{i, \mathbf{m}}^y &= \begin{cases} 1, & i = 0, y = h, g, f, e, \\ 0, & i > 0, y = h, \\ (-1)^i \prod_{p=1}^i (q^{n_2(m_1+(p-1)n_1)} - q^{n_1(m_2+(p-1)n_2)}), & i > 0, y = g, \\ \prod_{p=1}^i q^{n_1(m_2+(p-1)n_2)}, & i > 0, y = f, \\ (-1)^i \prod_{p=1}^i q^{n_2(m_1+(p-1)n_1)}, & i > 0, y = e, \end{cases} \\
\alpha_{y\mathbf{m}} &= \begin{cases} 0, & y = e, \\ q^{m_2 n_1}, & y = g, \\ -q^{m_1 n_2}, & y = h. \end{cases} \quad (1 - Et)^{\delta_{y,e}} = \begin{cases} 1 - Et, & y = e, \\ 1, & y = g, \\ 1, & y = h. \end{cases} \\
s_{\mathbf{m}} &= q^{n_2 m_1 + n_1 m_2 + n_1 n_2}.
\end{aligned}$$

The main result of this paper can be formulated as the following theorem.

Theorem 1.1. *There exist some noncommutative and noncocommutative Hopf algebra structures $(\mathcal{U}(\widehat{\mathfrak{sl}_2(\mathbb{C}_q)})[[t]], \mu, \tau, \Delta, \epsilon, S)$ on $\mathcal{U}(\widehat{\mathfrak{sl}_2(\mathbb{C}_q)})[[t]]$ over $\mathbb{C}[[t]]$, which preserve the product and the counit of $\mathcal{U}(\widehat{\mathfrak{sl}_2(\mathbb{C}_q)})[[t]]$, admitting the following corresponding coproducts and antipodes:*

(1) For $T = \sum_{i=1}^2 x_i d_i$, $E = g_{\mathbf{n}}$ with $[T, E] = E$ and $r = x_1 m_1 + x_2 m_2$, $y = e, f, g, h$, $i = 1, 2$,

$$\Delta(y_{\mathbf{m}}) = y_{\mathbf{m}} \otimes (1 - Et)^r + \sum_{j=0}^{\infty} \frac{\gamma_{j, \mathbf{m}}^y}{j!} T^{<j>} \otimes (1 - Et)^{-j} y_{\mathbf{m} + j\mathbf{n}} t^j,$$

$$S(y_{\mathbf{m}}) = \sum_{j=0}^{\infty} \frac{(-1)^{j+1} \gamma_{y_{\mathbf{m}}}^j}{j!} (1 - Et)^{-r} y_{\mathbf{m} + j\mathbf{n}} T_{1-c}^{<j>} t^j,$$

$$\Delta(d_i) = d_i \otimes 1 + 1 \otimes d_i - n_i T \otimes 1 + n_i T \otimes (1 - Et)^{-1},$$

$$\Delta(d) = d \otimes 1 + 1 \otimes d, \quad S(d) = -d, \quad S(d_i) = -d_i + n_i T E t.$$

(2) For $T = \sum_{i=1}^2 x_i d_i$, $E = e_{\mathbf{n}}$, with $[T, E] = E$ and $r = x_1 m_1 + x_2 m_2$, $y = e, g, h$, $i = 1, 2$,

$$\Delta(y_{\mathbf{m}}) = y_{\mathbf{m}} \otimes (1 - Et)^r + 1 \otimes y_{\mathbf{m}} + \alpha_{y_{\mathbf{m}}} T \otimes (1 - Et)^{-1} e_{\mathbf{m} + \mathbf{n}} t,$$

$$\Delta(f_{\mathbf{m}}) = \begin{cases} q^{m_2 n_1} T \otimes (1 - Et)^{-1} h_{\mathbf{m} + \mathbf{n}} t - q^{m_1 n_2} T \otimes (1 - Et)^{-1} g_{\mathbf{m} + \mathbf{n}} t \\ + f_{\mathbf{m}} \otimes (1 - Et)^r + 1 \otimes f_{\mathbf{m}} - s_{\mathbf{m}} T^{<2>} \otimes (1 - Et)^{-2} e_{\mathbf{m} + 2\mathbf{n}} t^2, & \mathbf{m} + \mathbf{n} \neq 0, \\ f_{-\mathbf{n}} \otimes (1 - Et)^{-1} + 1 \otimes f_{-\mathbf{n}} - q^{-n_2 n_1} T \otimes (1 - Et)^{-1} dt \\ - q^{-n_1 n_2} T^{<2>} \otimes (1 - Et)^{-2} E t^2, & \mathbf{m} + \mathbf{n} = 0, \end{cases}$$

$$\Delta(d) = d \otimes 1 + 1 \otimes d + 2T \otimes (1 - Et)^{-1} E t,$$

$$\Delta(d_i) = d_i \otimes 1 + 1 \otimes d_i - n_i T \otimes 1 + n_i T \otimes (1 - Et)^{-1},$$

$$S(y_{\mathbf{m}}) = -(1 - Et)^r y_{\mathbf{m}} + \alpha_{y_{\mathbf{m}}} (1 - Et)^r e_{\mathbf{m} + \mathbf{n}} T_1 t,$$

$$S(f_{\mathbf{m}}) = \begin{cases} q^{m_2 n_1} (1 - Et)^r h_{\mathbf{m} + \mathbf{n}} T_1 t - q^{m_1 n_2} (1 - Et)^r g_{\mathbf{m} + \mathbf{n}} T_1 t \\ - (1 - Et)^r f_{\mathbf{m}} + s_{\mathbf{m}} (1 - Et)^r e_{\mathbf{m} + 2\mathbf{n}} T_1^{<2>} t^2, & \mathbf{m} + \mathbf{n} \neq 0, \\ q^{-n_1 n_2} (1 - Et)^{-1} E T_1^{<2>} t^2 - (1 - Et)^{-1} f_{-\mathbf{n}} \\ - q^{-n_1 n_2} (1 - Et)^{-1} d T_1 t, & \mathbf{m} + \mathbf{n} = 0, \end{cases}$$

$$S(d) = -d + 2E T_1 t, \quad S(d_i) = -d_i + n_i T E t.$$

(3) For $T = \frac{1}{2}d$, $E = e_{\mathbf{n}}$ and $y = e, g, h$, $i = 1, 2$,

$$\Delta(y_{\mathbf{m}}) = y_{\mathbf{m}} \otimes (1 - Et)^{\delta_{y, e}} + 1 \otimes y_{\mathbf{m}} + \alpha_{y_{\mathbf{m}}} T \otimes (1 - Et)^{-1} e_{\mathbf{m} + \mathbf{n}} t,$$

$$\Delta(f_{\mathbf{m}}) = \begin{cases} q^{m_2 n_1} T \otimes (1 - Et)^{-1} h_{\mathbf{m}+\mathbf{n}} t - q^{m_1 n_2} T \otimes (1 - Et)^{-1} g_{\mathbf{m}+\mathbf{n}} t \\ + f_{\mathbf{m}} \otimes (1 - Et)^{-1} + 1 \otimes f_{\mathbf{m}} - s_{\mathbf{m}} T^{<2>} \otimes (1 - Et)^{-2} e_{\mathbf{m}+2\mathbf{n}} t^2, & \mathbf{m} + \mathbf{n} \neq 0, \\ f_{-\mathbf{n}} \otimes (1 - Et)^{-1} - q^{-n_2 n_1} T \otimes (1 - Et)^{-1} dt \\ + 1 \otimes f_{-\mathbf{n}} - q^{-n_1 n_2} T^{<2>} \otimes (1 - Et)^{-2} t^2, & \mathbf{m} + \mathbf{n} = 0, \end{cases}$$

$$\Delta(d) = d \otimes 1 + 1 \otimes d + 2T \otimes (1 - Et)^{-1} Et,$$

$$\Delta(d_i) = d_i \otimes 1 + 1 \otimes d_i + n_i T \otimes (1 - Et)^{-1} - n_i T \otimes 1,$$

$$S(y_{\mathbf{m}}) = -(1 - Et)^{\delta_{y,e}} y_{\mathbf{m}} + \alpha_{y_{\mathbf{m}}} e_{\mathbf{m}+\mathbf{n}} T_1 t,$$

$$S(f_{\mathbf{m}}) = \begin{cases} q^{m_2 n_1} (1 - Et)^{-1} h_{\mathbf{m}+\mathbf{n}} T_1 t - q^{m_1 n_2} (1 - Et)^{-1} T_1 g_{\mathbf{m}+\mathbf{n}} t \\ - (1 - Et)^{-1} f_{\mathbf{m}} + s_{\mathbf{m}} (1 - Et)^{-1} e_{\mathbf{m}+2\mathbf{n}} T_1^{<2>} t^2, & \mathbf{m} + \mathbf{n} \neq 0, \\ q^{-n_1 n_2} (1 - Et)^{-1} E T_1^{<2>} t^2 - (1 - Et)^{-1} f_{-\mathbf{n}} \\ - q^{-n_1 n_2} (1 - Et)^{-1} d T_1 t, & \mathbf{m} + \mathbf{n} = 0, \end{cases}$$

$$S(d) = -d + 2ET_1 t, \quad S(d_i) = -d_i + n_i T E t.$$

For $\mathbf{m} = (m_1, m_2)$, $\mathbf{n} = (n_1, n_2)$, $i \in \mathbb{Z}$, introduce the following notations that will be referred to in the following corollary:

$$\eta_{i,\mathbf{m}}^y = \begin{cases} 1, & i = 0, y = h, g, f, e, \\ 0, & i > 0, y = g, \\ (-1)^i \prod_{p=1}^i (q^{n_2(m_1+(p-1)n_1)} - q^{n_1(m_2+(p-1)n_2)}), & i > 0, y = h, \\ \prod_{p=1}^i q^{n_1(m_2+(p-1)n_2)}, & i > 0, y = e, \\ (-1)^i \prod_{p=1}^i q^{n_2(m_1+(p-1)n_1)}, & i > 0, y = f, \end{cases}$$

$$\beta_{y_{\mathbf{m}}} = \begin{cases} 0, & y = f, \\ -q^{m_1 n_2}, & y = g, \\ q^{m_2 n_1}, & y = h, \end{cases} \quad (1 - Et)^{\delta_{y,f}} = \begin{cases} 1 - Et, & y = f, \\ 1, & y = g, \\ 1, & y = h. \end{cases}$$

Combining Theorem 1.1 and the following involution of $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$:

$$\tau : e_{\mathbf{m}} \leftrightarrow f_{\mathbf{m}}, \quad g_{\mathbf{n}} \leftrightarrow h_{\mathbf{n}}, \quad d \leftrightarrow -d, \quad d_i \leftrightarrow d_i, \quad \forall \mathbf{m} \in \mathbf{Z}, \mathbf{n} \in \mathbf{Z}^*, i = 1, 2,$$

we can immediately derive the following corollary, which presents other three quantizations of $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$.

Corollary 1.2. *There exist some noncommutative and noncocommutative Hopf algebra structures $(\mathcal{U}(\widehat{\mathfrak{sl}_2(\mathbb{C}_q)})[[t]], \mu, \tau, \Delta, \epsilon, S)$ on $\mathcal{U}(\widehat{\mathfrak{sl}_2(\mathbb{C}_q)})[[t]]$ over $\mathbb{C}[[t]]$, which preserve the product and the counit of $\mathcal{U}(\widehat{\mathfrak{sl}_2(\mathbb{C}_q)})[[t]]$, admitting the following corresponding coproducts and antipodes:*

(1) For $T = \sum_{i=1}^2 x_i d_i$, $E = h_{\mathbf{n}}$ with $[T, E] = E$ and $r = x_1 m_1 + x_2 m_2$, $y = e, f, h, g$, $i = 1, 2$,

$$\Delta(y_{\mathbf{m}}) = y_{\mathbf{m}} \otimes (1 - Et)^r + \sum_{j=0}^{\infty} \frac{\eta_{j, \mathbf{m}}^y}{j!} T^{<j>} \otimes (1 - Et)^{-j} y_{\mathbf{m} + j\mathbf{n}} t^j,$$

$$S(y_{\mathbf{m}}) = \sum_{j=0}^{\infty} \frac{(-1)^{j+1} \eta_{j, \mathbf{m}}^y}{j!} (1 - Et)^{-r} y_{\mathbf{m} + j\mathbf{n}} T_{1-c}^{<j>} t^j,$$

$$\Delta(d_i) = d_i \otimes 1 + 1 \otimes d_i - n_i T \otimes 1 + n_i T \otimes (1 - Et)^{-1},$$

$$\Delta(d) = d \otimes 1 + 1 \otimes d, \quad S(d) = -d, \quad S(d_i) = -d_i + n_i T E t.$$

(2) For $T = \sum_{i=1}^2 x_i d_i$, $E = f_{\mathbf{n}}$, with $[T, E] = E$ and $r = x_1 m_1 + x_2 m_2$, $y = f, g, h$, $i = 1, 2$,

$$\Delta(y_{\mathbf{m}}) = y_{\mathbf{m}} \otimes (1 - Et)^r + 1 \otimes y_{\mathbf{m}} + \beta_{y_{\mathbf{m}}} T \otimes (1 - Et)^{-1} f_{\mathbf{m} + \mathbf{n}} t,$$

$$\Delta(e_{\mathbf{m}}) = \begin{cases} q^{m_2 n_1} T \otimes (1 - Et)^{-1} g_{\mathbf{m} + \mathbf{n}} t - q^{m_1 n_2} T \otimes (1 - Et)^{-1} h_{\mathbf{m} + \mathbf{n}} t \\ + e_{\mathbf{m}} \otimes (1 - Et)^r + 1 \otimes e_{\mathbf{m}} - s_{\mathbf{m}} T^{<2>} \otimes (1 - Et)^{-2} f_{\mathbf{m} + 2\mathbf{n}} t^2, & \mathbf{m} + \mathbf{n} \neq 0, \\ e_{-\mathbf{n}} \otimes (1 - Et)^{-1} + 1 \otimes e_{-\mathbf{n}} + q^{-n_2 n_1} T \otimes (1 - Et)^{-1} dt \\ - q^{-n_1 n_2} T^{<2>} \otimes (1 - Et)^{-2} E t^2, & \mathbf{m} + \mathbf{n} = 0, \end{cases}$$

$$\Delta(d) = d \otimes 1 + 1 \otimes d - 2T \otimes (1 - Et)^{-1} E t,$$

$$\Delta(d_i) = d_i \otimes 1 + 1 \otimes d_i - n_i T \otimes 1 + n_i T \otimes (1 - Et)^{-1},$$

$$S(y_{\mathbf{m}}) = -(1 - Et)^r y_{\mathbf{m}} + \beta_{y_{\mathbf{m}}} (1 - Et)^r f_{\mathbf{m} + \mathbf{n}} T_1 t,$$

$$S(e_{\mathbf{m}}) = \begin{cases} q^{m_2 n_1} (1 - Et)^r g_{\mathbf{m} + \mathbf{n}} T_1 t - q^{m_1 n_2} (1 - Et)^r h_{\mathbf{m} + \mathbf{n}} T_1 t \\ - (1 - Et)^r e_{\mathbf{m}} + s_{\mathbf{m}} (1 - Et)^r f_{\mathbf{m} + 2\mathbf{n}} T_1^{<2>} t^2, & \mathbf{m} + \mathbf{n} \neq 0, \\ q^{-n_1 n_2} (1 - Et)^{-1} E T_1^{<2>} t^2 - (1 - Et)^{-1} e_{-\mathbf{n}} \\ + q^{-n_1 n_2} (1 - Et)^{-1} d T_1 t, & \mathbf{m} + \mathbf{n} = 0, \end{cases}$$

$$S(d) = -d - 2E T_1 t, \quad S(d_i) = -d_i + n_i T E t.$$

(3) For $T = \frac{1}{2}d$, $E = f_{\mathbf{n}}$ and $y = f, g, h$, $i = 1, 2$,

$$\Delta(y_{\mathbf{m}}) = y_{\mathbf{m}} \otimes (1 - Et)^{\delta_{y, f}} + 1 \otimes y_{\mathbf{m}} + \beta_{y_{\mathbf{m}}} T \otimes (1 - Et)^{-1} e_{\mathbf{m} + \mathbf{n}} t,$$

$$\begin{aligned}
\Delta(e_{\mathbf{m}}) &= \begin{cases} q^{m_2 n_1} T \otimes (1 - Et)^{-1} g_{\mathbf{m}+\mathbf{n}} t - q^{m_1 n_2} T \otimes (1 - Et)^{-1} h_{\mathbf{m}+\mathbf{n}} t \\ + e_{\mathbf{m}} \otimes (1 - Et)^{-1} + 1 \otimes e_{\mathbf{m}} - s_{\mathbf{m}} T^{<2>} \otimes (1 - Et)^{-2} f_{\mathbf{m}+2\mathbf{n}} t^2, & \mathbf{m} + \mathbf{n} \neq 0, \\ e_{-\mathbf{n}} \otimes (1 - Et)^{-1} + q^{-n_2 n_1} T \otimes (1 - Et)^{-1} dt \\ + 1 \otimes e_{-\mathbf{n}} - q^{-n_1 n_2} T^{<2>} \otimes (1 - Et)^{-2} t^2, & \mathbf{m} + \mathbf{n} = 0, \end{cases} \\
\Delta(d) &= d \otimes 1 + 1 \otimes d - 2T \otimes (1 - Et)^{-1} Et, \\
\Delta(d_i) &= d_i \otimes 1 + 1 \otimes d_i + n_i T \otimes (1 - Et)^{-1} - n_i T \otimes 1, \\
S(y_{\mathbf{m}}) &= -(1 - Et)^{\delta_{y,f}} y_{\mathbf{m}} + \beta_{y_{\mathbf{m}}} f_{\mathbf{m}+\mathbf{n}} T_1 t, \\
S(e_{\mathbf{m}}) &= \begin{cases} q^{m_2 n_1} (1 - Et)^{-1} g_{\mathbf{m}+\mathbf{n}} T_1 t - q^{m_1 n_2} (1 - Et)^{-1} T_1 h_{\mathbf{m}+\mathbf{n}} t \\ - (1 - Et)^{-1} e_{\mathbf{m}} + s_{\mathbf{m}} (1 - Et)^{-1} f_{\mathbf{m}+2\mathbf{n}} T_1^{<2>} t^2, & \mathbf{m} + \mathbf{n} \neq 0, \\ q^{-n_1 n_2} (1 - Et)^{-1} E T_1^{<2>} t^2 - (1 - Et)^{-1} f_{-\mathbf{n}} \\ + q^{-n_1 n_2} (1 - Et)^{-1} d T_1 t, & \mathbf{m} + \mathbf{n} = 0, \end{cases} \\
S(d) &= -d - 2E T_1 t, \quad S(d_i) = -d_i + n_i T E t.
\end{aligned}$$

Convention 1.3. If an undefined term appears in an expression, we always treat it as zero, e.g., $g_0 = h_0 = 0$.

Remark 1.4. (1) We have in fact exhausted all the possibilities of Drinfel'd twist quantizations based on the “usual” noncommutative 2-dimensional Lie subalgebras $\{T, E\}$ of $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$ up to scalar multiplications (the “usual” means that one of the two elements, i.e., T , is in the Cartan subalgebra of $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$). This is also the main reason why we present 6 quantizations above. (We are currently engaging in an investigation of the Drinfel'd twist quantizations based on “unusual” choices of noncommutative 2-dimensional Lie subalgebras of $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$. However, it seems to us that heavy difficulties appear and that new techniques should be introduced during the process of such attempt.)

(2) Although the Lie bialgebra structures on the affine Lie algebra $\mathfrak{sl}_2(\mathbb{C}_q)$ have not been determined yet, we may obtain two quantizations of the affine Lie algebras $\mathfrak{sl}_2(\mathbb{C}_q)$ by restricting the third quantizations in Theorem 1.1 and Corollary 1.2 to $\mathfrak{sl}_2(\mathbb{C}_q)$ by taking $d_1 = d_2 = 0$.

2. Definition and preliminary results

We first recall some basic concepts and results based on a unital \mathbb{C} -algebra \mathcal{A} . For any element $x \in \mathcal{A}$, $a \in \mathbb{C}$, $r \in \mathbb{Z}_+$, set $x^{<r>} = x_0^{<r>}$, $x^{[r]} = x_0^{[r]}$, where

$$x_a^{<r>} = (x + a)(x + a + 1) \cdots (x + a + r - 1),$$

$$x_a^{[r]} = (x + a)(x + a - 1) \cdots (x + a - r + 1).$$

For convenient, set $x_a^{<0>} = x_a^{[0]} = 1$. The following lemma can be found in [17] or [18].

Lemma 2.1. *For any $x \in \mathcal{A}$, $a, d \in \mathbb{C}$ and $r, s, m \in \mathbb{Z}_+$, one has*

$$\begin{aligned} x_a^{<r+s>} &= x_a^{<r>} x_{a+r}^{<s>}, & x_a^{[r+s]} &= x_a^{[r]} x_{a-r}^{[s]}, & x_a^{[r]} &= x_{a-r+1}^{<r>}, \\ \sum_{r+s=m} \frac{(-1)^s}{r!s!} x_a^{[r]} x_d^{<s>} &= \binom{a-d}{m}, & \sum_{r+s=m} \frac{(-1)^s}{r!s!} x_a^{[r]} x_{d-r}^{[s]} &= \binom{a-d+m-1}{m}, \end{aligned} \quad (2.1)$$

where the binomial coefficient

$$\binom{a}{d} = \begin{cases} \frac{a(a-1)\cdots(a-d+1)}{d!}, & a \geq d, \\ 0, & a < d. \end{cases}$$

It is known that there is a natural Hopf algebra structure on the universal enveloping algebra of the Lie algebra $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$, denoted by $(\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}), \mu, \tau, \Delta_0, S_0, \epsilon_0)$ with

$$\Delta_0(L_{\mathbf{m}}) = L_{\mathbf{m}} \otimes 1 + 1 \otimes L_{\mathbf{m}}, \quad S_0(L_{\mathbf{m}}) = -L_{\mathbf{m}}, \quad \epsilon_0(L_{\mathbf{m}}) = 0, \quad \forall L_{\mathbf{m}} \in \widetilde{\mathfrak{L}}_{\mathbf{m}}.$$

Then a deformation of $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})$ is a topologically free $\mathbb{C}[[t]]$ -algebra $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})[[t]]$, i.e., it is an associative \mathbb{C} -algebra of formal power series with coefficients in $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})$ such that $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})[[t]]/t\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})[[t]] \cong \mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})$. Naturally, $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})[[t]]$ is equipped with a Hopf algebra structure induced from $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})$. We also denote it by $(\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})[[t]], \mu, \tau, \Delta_0, \epsilon_0, S_0)$.

Definition 2.2. Let $(\mathcal{A}, \mu, \tau, \Delta, S, \epsilon)$ be a Hopf algebra over a commutative ring. A *Drinfel'd twist* \mathcal{I} on \mathcal{A} is an invertible element of $\mathcal{A} \otimes \mathcal{A}$ such that

$$\begin{aligned} (\mathcal{I} \otimes 1)(\Delta \otimes Id)(\mathcal{I}) &= (1 \otimes \mathcal{I})(1 \otimes \Delta)(\mathcal{I}), \\ (\epsilon \otimes Id)(\mathcal{I}) &= 1 \otimes 1 = (Id \otimes \epsilon)(\mathcal{I}). \end{aligned}$$

It is known that the Drinfel'd twists play an important role in constructing a new Hopf algebra. We shall employ the following Lemma (see [8]) to complete the quantization of $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$ (also see [7, 21, 24]).

Lemma 2.3. *Let $(\mathcal{A}, \mu, \tau, \Delta_0, \epsilon_0, S_0)$ be a Hopf algebra over a commutative ring, \mathcal{I} a Drinfel'd twist on \mathcal{A} . Then*

$$(1) \quad u = \mu(Id \otimes S)(\mathcal{I}) \text{ is invertible in } \mathcal{A} \otimes \mathcal{A} \text{ with } u^{-1} = \mu(S \otimes Id)(\mathcal{I}).$$

$$(2) \quad \text{The algebra } (\mathcal{A}, \mu, \tau, \Delta, \epsilon, S) \text{ is a new Hopf algebra where}$$

$$\Delta = \mathcal{I}\Delta_0\mathcal{I}^{-1}, \quad \epsilon = \epsilon_0, \quad S = uS_0u^{-1}. \quad (2.2)$$

For formal variable t , and $c \in \mathbb{C}$, $T, E \in \widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$ with $[T, E] = E$, denote

$$\mathcal{I}_c = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_c^{[i]} \otimes E^i t^i, \quad \mathcal{J}_c = \mu(\text{Id} \otimes S_0)(\mathcal{I}_c),$$

$$I_c = \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes E^i t^i, \quad J_c = \mu(S_0 \otimes \text{Id})(I_c).$$

The following lemma also holds according to the corresponding lemma in [7, 21, 24].

Lemma 2.4.

$$\mathcal{J}_c = \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{[i]} E^i t^i, \quad J_c = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} E^i t^i, \quad (2.3)$$

$$\Delta_0(T^{[m]}) = \sum_{i=0}^m \binom{m}{i} T_{-c}^{[i]} \otimes T_c^{[m-i]}, \quad (2.4)$$

$$\mathcal{I}_c I_d = 1 \otimes (1 - Et)^{(c-d)}, \quad \mathcal{J}_c J_d = (1 - Et)^{-(c+d)}. \quad (2.5)$$

In particular, $\mathcal{I}_c, I_c, \mathcal{J}_c, J_c$ are invertible elements with $\mathcal{I}_c^{-1} = I_c$, $\mathcal{J}_c^{-1} = J_{-c}$ and \mathcal{I}_0 is a Drinfel'd twist of $(\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}))[[t]]$, $\mu, \tau, \Delta_0, \epsilon_0, S_0$.

3. Proof of Theorem 1.1 (1)

In this section, we take $T = x_1 d_1 + x_2 d_2$ for some $x_1, x_2 \in \mathbb{C}$, $\mathbf{n} = (n_1, n_2) \in \mathbf{Z}$ and $E = g_{\mathbf{n}}$ such that $[T, E] = E$. It is easy to see that $x_1 n_1 + x_2 n_2 = 1$. For $\mathbf{m} = (m_1, m_2) \in \mathbf{Z}$, $r = x_1 m_1 + x_2 m_2$, denote

$$\gamma_{i,\mathbf{m}}^g = (-1)^i \prod_{p=1}^i (q^{n_2(m_1+(p-1)n_1)} - q^{n_1(m_2+(p-1)n_2)}),$$

$$\gamma_{i,\mathbf{m}}^f = \prod_{p=1}^i q^{n_1(m_2+(p-1)n_2)}, \quad \gamma_{i,\mathbf{m}}^e = (-1)^i \prod_{p=1}^i q^{n_2(m_1+(p-1)n_1)}.$$

Lemma 3.1. *The following identities hold in $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})$ (where $l_{\mathbf{m}} \in \widetilde{\mathfrak{L}}_{\mathbf{m}}$):*

$$l_{\mathbf{m}} T_c^{[i]} = T_{c-r}^{[i]} l_{\mathbf{m}}, \quad l_{\mathbf{m}} T_c^{<i>} = T_{c-r}^{<i>} l_{\mathbf{m}}, \quad (3.1)$$

$$ET_c^{[i]} = T_{c-1}^{[i]} E, \quad ET_c^{<i>} = T_{c-1}^{<i>} E, \quad (3.2)$$

$$f_{\mathbf{m}} E^j = \sum_{i=0}^j \binom{j}{i} \gamma_{i,\mathbf{m}}^f E^{j-i} f_{\mathbf{m}+\mathbf{i}\mathbf{n}}, \quad (3.3)$$

$$e_{\mathbf{m}} E^j = \sum_{i=0}^j \binom{j}{i} \gamma_{i,\mathbf{m}}^e E^{j-i} e_{\mathbf{m}+\mathbf{i}\mathbf{n}}, \quad (3.4)$$

$$g_{\mathbf{m}} E^j = \sum_{i=0}^j \binom{j}{i} \gamma_{i,\mathbf{m}}^g E^{j-i} g_{\mathbf{m}+\mathbf{i}\mathbf{n}}, \quad (3.5)$$

$$dE^j = E^j d, \quad h_{\mathbf{m}} E^j = E^j h_{\mathbf{m}}, \quad (3.6)$$

$$d_1 E^j = E^j d_1 + j n_1 E^j, \quad d_2 E^j = E^j d_2 + j n_2 E^j. \quad (3.7)$$

Proof. Since $[T, l_{\mathbf{m}}] = (x_1 m_1 + x_2 m_2) l_{\mathbf{m}} = r l_{\mathbf{m}}$, $[d_1, E^j] = j n_1 E^j$ and $[d_2, E^j] = j n_2 E^j$, we obtain equations (3.1) and (3.7). Equation (3.2) is a special case of (3.1). Equations (3.6) are obtained by $[d, E] = [h, E] = 0$.

Using induction on i , one has

$$(adE)^i e_{\mathbf{m}} = \prod_{p=1}^i q^{n_2(m_1+(p-1)n_1)} e_{\mathbf{m}+i\mathbf{n}} = (-1)^i \gamma_{i,\mathbf{m}}^e e_{\mathbf{m}+i\mathbf{n}},$$

$$(adE)^i f_{\mathbf{m}} = (-1)^i \prod_{p=1}^i q^{n_1(m_2+(p-1)n_2)} f_{\mathbf{m}+i\mathbf{n}} = (-1)^i \gamma_{i,\mathbf{m}}^f f_{\mathbf{m}+i\mathbf{n}},$$

$$(adE)^i g_{\mathbf{m}} = \prod_{p=1}^i (q^{n_2(m_1+(p-1)n_1)} - q^{n_1(m_2+(p-1)n_2)}) g_{\mathbf{m}+i\mathbf{n}} = (-1)^i \gamma_{i,\mathbf{m}}^g g_{\mathbf{m}+i\mathbf{n}}.$$

Then, we obtain the equations (3.3), (3.4), (3.5) as follows:

$$f_{\mathbf{m}} E^j = \sum_{i=0}^j (-1)^i \binom{j}{i} E^{j-i} (adE)^i (f_{\mathbf{m}}) = \sum_{i=0}^j \binom{j}{i} \gamma_{i,\mathbf{m}}^f E^{j-i} f_{\mathbf{m}+i\mathbf{n}},$$

$$e_{\mathbf{m}} E^j = \sum_{i=0}^j (-1)^i \binom{j}{i} E^{j-i} (adE)^i (e_{\mathbf{m}}) = \sum_{i=0}^j \binom{j}{i} \gamma_{i,\mathbf{m}}^e E^{j-i} e_{\mathbf{m}+i\mathbf{n}},$$

$$g_{\mathbf{m}} E^j = \sum_{i=0}^j (-1)^i \binom{j}{i} E^{j-i} (adE)^i (g_{\mathbf{m}}) = \sum_{i=0}^j \binom{j}{i} \gamma_{i,\mathbf{m}}^g E^{j-i} g_{\mathbf{m}+i\mathbf{n}}.$$

□

Lemma 3.2. *The following identities hold in $\mathcal{U}(\widehat{\mathfrak{sl}_2(\mathbb{C}_q)})$ (where $l_{\mathbf{m}} \in \tilde{\mathfrak{L}}_{\mathbf{m}}$):*

$$(l_{\mathbf{m}} \otimes 1) I_c = I_{c-r} (l_{\mathbf{m}} \otimes 1), \quad (3.8)$$

$$(1 \otimes d_1) I_c = n_1 I_{c+1} (T_c \otimes Et) + I_c (1 \otimes d_1), \quad (3.9)$$

$$(1 \otimes d_2) I_c = n_2 I_{c+1} (T_c \otimes Et) + I_c (1 \otimes d_2), \quad (3.10)$$

$$(1 \otimes h_{\mathbf{m}}) I_c = I_c (1 \otimes h_{\mathbf{m}}), \quad (1 \otimes d) I_c = I_c (1 \otimes d), \quad (3.11)$$

$$(1 \otimes f_{\mathbf{m}}) I_c = \sum_{i=0}^{\infty} \frac{\gamma_{i,\mathbf{m}}^f}{i!} I_{c+i} (T_c^{<i>} \otimes f_{\mathbf{m}+i\mathbf{n}} t^i), \quad (3.12)$$

$$(1 \otimes e_{\mathbf{m}}) I_c = \sum_{i=0}^{\infty} \frac{\gamma_{i,\mathbf{m}}^e}{i!} I_{c+i} (T_c^{<i>} \otimes e_{\mathbf{m}+i\mathbf{n}} t^i), \quad (3.13)$$

$$(1 \otimes g_{\mathbf{m}}) I_c = \sum_{i=0}^{\infty} \frac{\gamma_{i,\mathbf{m}}^g}{i!} I_{c+i} (T_c^{<i>} \otimes g_{\mathbf{m}+i\mathbf{n}} t^i). \quad (3.14)$$

Proof. For $l_{\mathbf{m}} \in \widetilde{\mathfrak{L}}_{\mathbf{m}}$, using formula (3.1),

$$\begin{aligned} (l_{\mathbf{m}} \otimes 1)I_c &= (l_{\mathbf{m}} \otimes 1)\left(\sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes E^i t^i\right) \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} T_{c-r}^{<i>} l_{\mathbf{m}} \otimes E^i t^i = I_{c-r}(l_{\mathbf{m}} \otimes 1). \end{aligned}$$

Hence, we obtain equation (3.8). It is obvious that $d_1 E^i = [d_1, E^i] + E^i d_1 = in_1 E^i + E^i d_1$ and $d_2 E^i = [d_2, E^i] + E^i d_2 = in_2 E^i + E^i d_2$, which mean

$$\begin{aligned} (1 \otimes d_1)I_c &= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes d_1 E^i t^i = \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes (in_1 E^i + E^i d_1) t^i \\ &= \sum_{i=0}^{\infty} \frac{n_1}{i!} T_c^{<i+1>} \otimes E^{i+1} t^{i+1} + \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes E^i d_1 t^i \\ &= \sum_{i=0}^{\infty} \frac{n_1}{i!} T_c T_{c+1}^{<i>} \otimes E^{i+1} t^{i+1} + I_c(1 \otimes d_1) \\ &= n_1 I_{c+1}(T_c \otimes Et) + I_c(1 \otimes d_1), \\ (1 \otimes d_2)I_c &= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes d_2 E^i t^i = \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes (in_2 E^i + E^i d_2) t^i \\ &= \sum_{i=0}^{\infty} \frac{n_2}{i!} T_c^{<i+1>} \otimes E^{i+1} t^{i+1} + \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes E^i d_2 t^i \\ &= \sum_{i=0}^{\infty} \frac{n_2}{i!} T_c T_{c+1}^{<i>} \otimes E^{i+1} t^{i+1} + I_c(1 \otimes d_2) \\ &= n_2 I_{c+1}(T_c \otimes Et) + I_c(1 \otimes d_2). \end{aligned}$$

Hence, we complete the proof of equations (3.9), (3.10) respectively.

Since $[h_{\mathbf{m}}, E] = [d, E] = 0$, for all $h_{\mathbf{m}} \in \mathcal{H}_{\mathbf{m}}$, equation (3.11) is obviously established.

Using formula (2.1), (3.3), we obtain equations (3.12) as follows,

$$\begin{aligned} (1 \otimes f_{\mathbf{m}})I_c &= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes f_{\mathbf{m}} E^i t^i \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes \left(\sum_{j=0}^i \binom{i}{j} \gamma_{j,\mathbf{m}}^f E^{i-j} f_{\mathbf{m}+j\mathbf{n}}\right) t^i \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(i+j)!} \binom{i+j}{j} \gamma_{j,\mathbf{m}}^f T_c^{<i+j>} \otimes E^i f_{\mathbf{m}+j\mathbf{n}} t^{i+j} \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \gamma_{j,\mathbf{m}}^f T_c^{<j>} \sum_{i=0}^{\infty} \frac{1}{i!} T_{c+j}^{<i>} \otimes E^i f_{\mathbf{m}+j\mathbf{n}} t^{i+j} \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \gamma_{j,\mathbf{m}}^f I_{c+j}(T_c^{<j>} \otimes f_{\mathbf{m}+j\mathbf{n}} t^j). \end{aligned}$$

Similarly, equations (3.13) and (3.14) are also tenable. Now, we complete the proof of this lemma. \square

Lemma 3.3. *The following identities hold in $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})$:*

$$\begin{aligned} h_{\mathbf{m}}J_c &= J_{c+r}h_{\mathbf{m}}, \quad dJ_c = J_cd, \\ d_1J_c &= J_cd_1 - n_1J_cT_{-c}Et, \\ d_2J_c &= J_cd_2 - n_2J_cT_{-c}Et, \\ f_{\mathbf{m}}J_c &= J_{c+r}\left(\sum_{j=0}^{\infty} \frac{(-1)^j \gamma_{j,\mathbf{m}}^f}{j!} f_{\mathbf{m}+j\mathbf{n}} T_{1-c}^{<j>} t^j\right), \\ e_{\mathbf{m}}J_c &= J_{c+r}\left(\sum_{j=0}^{\infty} \frac{(-1)^j \gamma_{j,\mathbf{m}}^e}{j!} e_{\mathbf{m}+j\mathbf{n}} T_{1-c}^{<j>} t^j\right), \\ g_{\mathbf{m}}J_c &= J_{c+r}\left(\sum_{j=0}^{\infty} \frac{(-1)^j \gamma_{j,\mathbf{m}}^g}{j!} g_{\mathbf{m}+j\mathbf{n}} T_{1-c}^{<j>} t^j\right). \end{aligned}$$

Proof. Using the formulae (2.1), (2.3), (3.1) and (3.6), we have

$$h_{\mathbf{m}}J_c = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} h_{\mathbf{m}} T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} h_{\mathbf{m}} E^i t^i = J_{c+r} h_{\mathbf{m}}.$$

Similarly, $dJ_c = J_cd$. Since formulas (2.1), (2.3), (3.1) and (3.7), there are

$$\begin{aligned} d_1J_c &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} d_1 T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} d_1 E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} (in_1 E^i + E^i d_1) t^i \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} E^i d_1 t^i + \sum_{i=0}^{\infty} \frac{(-1)^{i+1} n_1}{i!} T_{-c}^{[i]} T_{-c-i} E^{i+1} t^{i+1} = J_cd_1 - n_1 J_c T_{-c} Et, \\ d_2J_c &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} d_2 T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} d_2 E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} (in_2 E^i + E^i d_2) t^i \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} E^i d_2 t^i + \sum_{i=0}^{\infty} \frac{(-1)^{i+1} n_2}{i!} T_{-c}^{[i]} T_{-c-i} E^{i+1} t^{i+1} = J_cd_2 - n_2 J_c T_{-c} Et. \end{aligned}$$

The last three equations could be obtained by the formulas (2.1), (2.3) and formulas from (3.1) to (3.5). For symbol $y_{\mathbf{m}} = f_{\mathbf{m}}, e_{\mathbf{m}}$ or $g_{\mathbf{m}}$, there is

$$\begin{aligned} y_{\mathbf{m}}J_c &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} y_{\mathbf{m}} T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} y_{\mathbf{m}} E^i t^i \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} \left(\sum_{j=0}^i \binom{i}{j} \gamma_{j,\mathbf{m}}^y E^{i-j} y_{\mathbf{m}+j\mathbf{n}} \right) t^i \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{i! j!} \gamma_{j,\mathbf{m}}^y T_{-c-r}^{[i]} T_{-c-r-i}^{[j]} E^i y_{\mathbf{m}+j\mathbf{n}} t^{i+j} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} E^i t^i \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \gamma_{j,\mathbf{m}}^y y_{\mathbf{m}+j\mathbf{n}} T_{-c+j}^{[j]} t^j \\
&= J_{c+r} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \gamma_{j,\mathbf{m}}^y y_{\mathbf{m}+j\mathbf{n}} T_{1-c}^{<j>} t^j.
\end{aligned}$$

□

Proof of Theorem 1.1 (1). Using equations (2.2), (2.5) and all the lemmas above, for symbol $y_{\mathbf{m}} = f_{\mathbf{m}}, e_{\mathbf{m}}$ or $g_{\mathbf{m}}$, we obtain

$$\begin{aligned}
\Delta(y_{\mathbf{m}}) &= \mathcal{I}\Delta_0(y_{\mathbf{m}})\mathcal{I}^{-1} = \mathcal{I}(y_{\mathbf{m}} \otimes 1 + 1 \otimes y_{\mathbf{m}})I \\
&= \mathcal{I}I_{-r}(y_{\mathbf{m}} \otimes 1) + \mathcal{I}\left(\sum_{i=0}^{\infty} \frac{\gamma_{i,\mathbf{m}}^y}{i!} I_i(T^{<i>} \otimes y_{\mathbf{m}+i\mathbf{n}} t^i)\right) \\
&= (1 \otimes (1 - Et)^r)(y_{\mathbf{m}} \otimes 1) + \sum_{i=0}^{\infty} \frac{\gamma_{i,\mathbf{m}}^y}{i!} (1 \otimes (1 - Et)^{-i})(T^{<i>} \otimes y_{\mathbf{m}+i\mathbf{n}} t^i) \\
&= y_{\mathbf{m}} \otimes (1 - Et)^r + \sum_{i=0}^{\infty} \frac{\gamma_{i,\mathbf{m}}^y}{i!} T^{<i>} \otimes (1 - Et)^{-i} y_{\mathbf{m}+i\mathbf{n}} t^i, \\
\Delta(h_{\mathbf{m}}) &= \mathcal{I}\Delta_0(h_{\mathbf{m}})\mathcal{I}^{-1} = \mathcal{I}(h_{\mathbf{m}} \otimes 1 + 1 \otimes h_{\mathbf{m}})I \\
&= \mathcal{I}I_{-r}(h_{\mathbf{m}} \otimes 1) + \mathcal{I}I(1 \otimes h_{\mathbf{m}}) = h_{\mathbf{m}} \otimes (1 - Et)^r + 1 \otimes h_{\mathbf{m}}, \\
\Delta(d) &= \mathcal{I}\Delta_0(d)\mathcal{I}^{-1} = \mathcal{I}(d \otimes 1 + 1 \otimes d)I = d \otimes 1 + 1 \otimes d, \\
\Delta(d_1) &= \mathcal{I}\Delta_0(d_1)\mathcal{I}^{-1} = \mathcal{I}(d_1 \otimes 1 + 1 \otimes d_1)I \\
&= \mathcal{I}I(d_1 \otimes 1) + \mathcal{I}(n_1 I_1(T \otimes Et) + I(1 \otimes d_1)) \\
&= d_1 \otimes 1 + 1 \otimes d_1 + n_1 T \otimes (1 - Et)^{-1} Et \\
&= d_1 \otimes 1 + 1 \otimes d_1 - n_1 T \otimes 1 + n_1 T \otimes (1 - Et)^{-1}, \\
\Delta(d_2) &= \mathcal{I}\Delta_0(d_2)\mathcal{I}^{-1} = \mathcal{I}(d_2 \otimes 1 + 1 \otimes d_2)I \\
&= \mathcal{I}I(d_2 \otimes 1) + \mathcal{I}(n_2 I_1(T \otimes Et) + I(1 \otimes d_2)) \\
&= d_2 \otimes 1 + 1 \otimes d_2 + n_2 T \otimes (1 - Et)^{-1} Et \\
&= d_2 \otimes 1 + 1 \otimes d_2 - n_2 T \otimes 1 + n_2 T \otimes (1 - Et)^{-1}.
\end{aligned}$$

In addition, we also obtain,

$$\begin{aligned}
S(y_{\mathbf{m}}) &= \mathcal{J}s_0(y_{\mathbf{m}})J = -\mathcal{J}y_{\mathbf{m}}J = -\mathcal{J}\left(J_r \sum_{j=0}^{\infty} \frac{(-1)^j \gamma_{j,\mathbf{m}}^y}{j!} y_{\mathbf{m}+j\mathbf{n}} T_{1-c}^{<j>} t^j\right) \\
&= \sum_{j=0}^{\infty} \frac{(-1)^{j+1} \gamma_{j,\mathbf{m}}^y}{j!} (1 - Et)^{-r} y_{\mathbf{m}+j\mathbf{n}} T_{1-c}^{<j>} t^j, \\
S(h_{\mathbf{m}}) &= \mathcal{J}s_0(h_{\mathbf{m}})J = -\mathcal{J}h_{\mathbf{m}}J = -\mathcal{J}J_r h_{\mathbf{m}} = -(1 - Et)^{-r} h_{\mathbf{m}}, \\
S(d) &= \mathcal{J}s_0(d)J = -\mathcal{J}dJ = -d,
\end{aligned}$$

$$\begin{aligned}
S(d_1) &= \mathcal{J}s_0(d_1)J = -\mathcal{J}d_1J = -\mathcal{J}(Jd_1 - n_1JT Et) = -d_1 + n_1TEt, \\
S(d_2) &= \mathcal{J}s_0(d_2)J = -\mathcal{J}d_2J = -\mathcal{J}(Jd_2 - n_2JT Et) = -d_2 + n_2TEt.
\end{aligned}$$

□

4. Proof of Theorem 1.1(2)

In this section, we take $T = x_1d_1 + x_2d_2$ for some $x_1, x_2 \in \mathbb{C}$, $\mathbf{n} = (n_1, n_2) \in \mathbf{Z}$ and $E = e_{\mathbf{n}}$ such that $[T, E] = E$. It is easy to see that $x_1n_1 + x_2n_2 = 1$. The expressions only referring to T in Section 3 are also tenable in this section, such as expressions (3.1), (3.2), (3.8). For $\mathbf{m} = (m_1, m_2) \in \mathbf{Z}$, $r = x_1m_1 + x_2m_2$, denote

$$s_{\mathbf{m}} = q^{m_1n_2+m_2n_1+n_1n_2}.$$

Lemma 4.1. *The following identities hold in $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})$:*

$$e_{\mathbf{m}}E^j = E^je_{\mathbf{m}}, \quad dE^j = E^jd + 2jE^j, \quad (4.1)$$

$$d_1E^j = E^jd_1 + jn_1E^j, \quad d_2E^j = E^jd_2 + jn_2E^j, \quad (4.2)$$

$$f_{\mathbf{m}}E^j = \begin{cases} jq^{m_2n_1}E^{j-1}h_{\mathbf{m}+\mathbf{n}} - jq^{m_1n_2}E^{j-1}g_{\mathbf{m}+\mathbf{n}} \\ + E^jf_{\mathbf{m}} - 2s_{\mathbf{m}}\binom{j}{2}E^{j-2}e_{\mathbf{m}+2\mathbf{n}}, & \mathbf{m} + \mathbf{n} \neq 0, \\ E^jf_{-\mathbf{n}} - jq^{-n_1n_2}E^{j-1}d - 2\binom{j}{2}q^{-n_2n_1}E^{j-1}, & \mathbf{m} + \mathbf{n} = 0, \end{cases} \quad (4.3)$$

$$g_{\mathbf{m}}E^j = E^jg_{\mathbf{m}} + jq^{m_2n_1}E^{j-1}e_{\mathbf{m}+\mathbf{n}}, \quad (4.4)$$

$$h_{\mathbf{m}}E^j = E^jh_{\mathbf{m}} - jq^{m_1n_2}E^{j-1}e_{\mathbf{m}+\mathbf{n}}. \quad (4.5)$$

Proof. Equations (4.1) and (4.2) are obtained by formulas $[e_{\mathbf{m}}, E] = 0$, $[d, E] = 2E$ and $[d_1, E^j] = jn_1E^j$, $[d_2, E^j] = jn_2E^j$ respectively.

By the definition, if $\mathbf{m} + \mathbf{n} \neq 0$, there is

$$(adE)^i(f_{\mathbf{m}}) = (ade_{\mathbf{n}})^i(f_{\mathbf{m}}) = \begin{cases} f_{\mathbf{m}}, & i = 0, \\ q^{m_1n_2}g_{\mathbf{m}+\mathbf{n}} - q^{m_2n_1}h_{\mathbf{m}+\mathbf{n}}, & i = 1, \\ -2s_{\mathbf{m}}e_{\mathbf{m}+2\mathbf{n}}, & i = 2, \\ 0, & i > 2, \end{cases}$$

where $s_{\mathbf{m}} = q^{m_1n_2+m_2n_1+n_1n_2} \in \mathbb{C}$. Thus,

$$\begin{aligned}
f_{\mathbf{m}}E^j &= \sum_{i=0}^j (-1)^i \binom{j}{i} E^{j-i} (adE)^i(f_{\mathbf{m}}) \\
&= E^jf_{\mathbf{m}} - jE^{j-1}(q^{m_1n_2}g_{\mathbf{m}+\mathbf{n}} - q^{m_2n_1}h_{\mathbf{m}+\mathbf{n}}) + \binom{j}{2}E^{j-2}(-2s_{\mathbf{m}}e_{\mathbf{m}+2\mathbf{n}})
\end{aligned}$$

$$= E^j f_{\mathbf{m}} - jq^{m_1 n_2} E^{j-1} g_{\mathbf{m}+\mathbf{n}} + jq^{m_2 n_1} E^{j-1} h_{\mathbf{m}+\mathbf{n}} - 2s_{\mathbf{m}} \binom{j}{2} E^{j-2} e_{\mathbf{m}+2\mathbf{n}}.$$

Similarly, there are

$$(adE)^i(f_{-\mathbf{n}}) = (ade_{\mathbf{n}})^i(f_{-\mathbf{n}}) = \begin{cases} f_{-\mathbf{n}}, & i = 0, \\ q^{-n_1 n_2} d, & i = 1, \\ -2q^{-n_1 n_2} e_{\mathbf{n}}, & i = 2, \\ 0, & i > 2, \end{cases}$$

and

$$\begin{aligned} f_{-\mathbf{n}} E^j &= \sum_{i=0}^j (-1)^i \binom{j}{i} E^{j-i} (adE)^i(f_{-\mathbf{n}}) \\ &= E^j f_{-\mathbf{n}} - j E^{j-1} (q^{-n_1 n_2} d) + \binom{j}{2} E^{j-2} (-2q^{-n_1 n_2} E) \\ &= E^j f_{-\mathbf{n}} - jq^{-n_1 n_2} E^{j-1} d - 2q^{-n_1 n_2} \binom{j}{2} E^{j-1}. \end{aligned}$$

Furthermore, one can obtain

$$(adE)^i(g_{\mathbf{m}}) = (ade_{\mathbf{n}})^i(g_{\mathbf{m}}) = \begin{cases} g_{\mathbf{m}}, & i = 0, \\ -q^{m_2 n_1} e_{\mathbf{m}+\mathbf{n}}, & i = 1, \\ 0, & i \geq 2. \end{cases}$$

$$(adE)^i(h_{\mathbf{m}}) = (ade_{\mathbf{n}})^i(h_{\mathbf{m}}) = \begin{cases} h_{\mathbf{m}}, & i = 0, \\ q^{m_1 n_2} e_{\mathbf{m}+\mathbf{n}}, & i = 1, \\ 0, & i \geq 2. \end{cases}$$

These mean

$$\begin{aligned} g_{\mathbf{m}} E^j &= \sum_{i=0}^j (-1)^i \binom{j}{i} E^{j-i} (adE)^i(g_{\mathbf{m}}) = E^j g_{\mathbf{m}} + jq^{m_2 n_1} E^{j-1} e_{\mathbf{m}+\mathbf{n}}, \\ h_{\mathbf{m}} E^j &= \sum_{i=0}^j (-1)^i \binom{j}{i} E^{j-i} (adE)^i(h_{\mathbf{m}}) = E^j h_{\mathbf{m}} - jq^{m_1 n_2} E^{j-1} e_{\mathbf{m}+\mathbf{n}}. \end{aligned}$$

□

Lemma 4.2. *The following identities hold in $\mathcal{U}(\widehat{\mathfrak{sl}_2(\mathbb{C}_q)})$ (where $l_{\mathbf{m}} \in \widetilde{\mathfrak{L}}_{\mathbf{m}}$):*

$$(l_{\mathbf{m}} \otimes 1)I_c = I_{c-r}(l_{\mathbf{m}} \otimes 1), \quad (4.6)$$

$$(1 \otimes d_1)I_c = n_1 I_{c+1}(T_c \otimes Et) + I_c(1 \otimes d_1), \quad (4.7)$$

$$(1 \otimes d_2)I_c = n_2 I_{c+1}(T_c \otimes Et) + I_c(1 \otimes d_2), \quad (4.8)$$

$$(1 \otimes e_{\mathbf{m}})I_c = I_c(1 \otimes e_{\mathbf{m}}), \quad (4.9)$$

$$(1 \otimes f_{\mathbf{m}})I_c = \begin{cases} q^{m_2 n_1} I_{c+1}(T_c \otimes h_{\mathbf{m}+\mathbf{n}t}) - q^{m_1 n_2} I_{c+1}(T_c \otimes g_{\mathbf{m}+\mathbf{n}t}) \\ + I_c(1 \otimes f_{\mathbf{m}}) - s_{\mathbf{m}} I_{c+2}(T_c^{<2>} \otimes e_{\mathbf{m}+2\mathbf{n}t^2}), & \mathbf{m} + \mathbf{n} \neq 0, \\ I_c(1 \otimes f_{-\mathbf{n}}) - q^{-n_2 n_1} I_{c+1}(T_c \otimes dt) \\ - q^{-n_1 n_2} I_{c+2}(T_c^{<2>} \otimes Et^2), & \mathbf{m} + \mathbf{n} = 0, \end{cases} \quad (4.10)$$

$$(1 \otimes g_{\mathbf{m}})I_c = I_c(1 \otimes g_{\mathbf{m}}) + q^{m_2 n_1} I_{c+1}(T_c \otimes e_{\mathbf{m}+\mathbf{n}t}), \quad (4.11)$$

$$(1 \otimes h_{\mathbf{m}})I_c = I_c(1 \otimes h_{\mathbf{m}}) - q^{m_1 n_2} I_{c+1}(T_c \otimes e_{\mathbf{m}+\mathbf{n}t}), \quad (4.12)$$

$$(1 \otimes d)I_c = I_c(1 \otimes d) + 2I_{c+1}(T_c \otimes Et). \quad (4.13)$$

Proof. Equations (4.6), (4.7) and (4.8) are similar as (3.8), (3.9) and (3.10) in Lemma 3.2.

It is easy to obtain equation (4.9) by $[e_{\mathbf{m}}, E] = 0$.

Using formula (2.1), (4.3), we obtain equation (4.10) as follows, for $\mathbf{m} + \mathbf{n} \neq 0$,

$$\begin{aligned} (1 \otimes f_{\mathbf{m}})I_c &= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes f_{\mathbf{m}} E^i t^i = \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes \\ &\quad (E^i f_{\mathbf{m}} - i q^{n_2 m_1} E^{i-1} g_{\mathbf{m}+\mathbf{n}} + i q^{n_1 m_2} E^{i-1} h_{\mathbf{m}+\mathbf{n}} - 2 s_{\mathbf{m}} \binom{i}{2} E^{i-2} e_{\mathbf{m}+2\mathbf{n}}) t^i \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes E^i f_{\mathbf{m}} t^i - q^{n_2 m_1} \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i+1>} \otimes E^i g_{\mathbf{m}+\mathbf{n}} t^{i+1} \\ &\quad + q^{n_1 m_2} \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i+1>} \otimes E^i h_{\mathbf{m}+\mathbf{n}} t^{i+1} - s_{\mathbf{m}} \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i+2>} \otimes E^i e_{\mathbf{m}+2\mathbf{n}} t^{i+2} \\ &= I_c(1 \otimes f_{\mathbf{m}}) - q^{n_2 m_1} I_{c+1}(T_c \otimes g_{\mathbf{m}+\mathbf{n}} t) + q^{n_1 m_2} I_{c+1}(T_c \otimes h_{\mathbf{m}+\mathbf{n}} t) \\ &\quad - s_{\mathbf{m}} I_{c+2}(T_c^{<2>} \otimes e_{\mathbf{m}+2\mathbf{n}} t^2). \end{aligned}$$

Similarly

$$\begin{aligned} (1 \otimes f_{-\mathbf{n}})I_c &= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes f_{-\mathbf{n}} E^i t^i \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes (E^i f_{-\mathbf{n}} - i q^{-n_2 n_1} E^{i-1} d - 2 q^{-n_2 n_1} \binom{i}{2} E^{i-1}) t^i \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes E^i f_{-\mathbf{n}} t^i - q^{-n_2 n_1} \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i+1>} \otimes E^i d t^{i+1} \\ &\quad - q^{-n_1 n_2} \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i+2>} \otimes E^{i+1} t^{i+2} \\ &= I_c(1 \otimes f_{-\mathbf{n}}) - q^{-n_2 n_1} I_{c+1}(T_c \otimes dt) - q^{-n_1 n_2} I_{c+2}(T_c^{<2>} \otimes Et^2). \end{aligned}$$

Moreover, we also have

$$\begin{aligned}
(1 \otimes g_{\mathbf{m}})I_c &= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes g_{\mathbf{m}} E^i t^i = \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes (E^i g_{\mathbf{m}} + i q^{m_2 n_1} E^{i-1} e_{\mathbf{m}+\mathbf{n}}) t^i \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes E^i g_{\mathbf{m}} t^i + q^{m_2 n_1} \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i+1>} \otimes E^i e_{\mathbf{m}+\mathbf{n}} t^{i+1} \\
&= I_c(1 \otimes g_{\mathbf{m}}) + q^{m_2 n_1} I_{c+1}(T_c \otimes e_{\mathbf{m}+\mathbf{n}} t), \\
(1 \otimes h_{\mathbf{m}})I_c &= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes h_{\mathbf{m}} E^i t^i = \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes (E^i h_{\mathbf{m}} - i q^{m_1 n_2} E^{i-1} e_{\mathbf{m}+\mathbf{n}}) t^i \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes E^i h_{\mathbf{m}} t^i - q^{m_1 n_2} \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i+1>} \otimes E^i e_{\mathbf{m}+\mathbf{n}} t^{i+1} \\
&= I_c(1 \otimes h_{\mathbf{m}}) - q^{m_1 n_2} I_{c+1}(T_c \otimes e_{\mathbf{m}+\mathbf{n}} t), \\
(1 \otimes d)I_c &= \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes d E^i t^i = \sum_{i=0}^{\infty} \frac{1}{i!} T_c^{<i>} \otimes (E^i d + 2i E^i) t^i \\
&= I_c(1 \otimes d) + 2I_{c+1}(T_c \otimes E t).
\end{aligned}$$

□

Lemma 4.3. *The following identities hold in $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})$:*

$$e_{\mathbf{m}} J_c = J_{c+r} e_{\mathbf{m}}, \quad (4.14)$$

$$d_1 J_c = J_c d_1 - n_1 J_c T_{-c} E t, \quad (4.15)$$

$$d_2 J_c = J_c d_2 - n_2 J_c T_{-c} E t, \quad (4.16)$$

$$f_{\mathbf{m}} J_c = \begin{cases} q^{m_1 n_2} J_{c+r} (T_{-c-r} g_{\mathbf{m}+\mathbf{n}} t) - q^{m_2 n_1} J_{c+r} (h_{\mathbf{m}+\mathbf{n}} T_{1-c} t) \\ + J_{c+r} f_{\mathbf{m}} - s_{\mathbf{m}} J_{c+r} (e_{\mathbf{m}+2\mathbf{n}} T_{1-c}^{<2>} t^2), & \mathbf{m} + \mathbf{n} \neq 0, \\ J_{c-1} f_{-\mathbf{n}} + q^{n_1 n_2} J_{c-1} (d T_{1-c} t) - q^{n_1 n_2} J_{c-1} E T_{1-c}^{<2>} t^2, & \mathbf{m} + \mathbf{n} = 0, \end{cases} \quad (4.17)$$

$$g_{\mathbf{m}} J_c = J_{c+r} g_{\mathbf{m}} - q^{n_1 m_2} J_{c+r} (e_{\mathbf{m}+\mathbf{n}} T_{1-c} t), \quad (4.18)$$

$$h_{\mathbf{m}} J_c = J_{c+r} h_{\mathbf{m}} + q^{n_2 m_1} J_{c+r} (e_{\mathbf{m}+\mathbf{n}} T_{1-c} t), \quad (4.19)$$

$$d J_c = J_c d - 2 J_c (E T_{1-c} t). \quad (4.20)$$

Proof. Using the formulas (2.3), (3.1) and (4.1), there is

$$\begin{aligned}
e_{\mathbf{m}} J_c &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} e_{\mathbf{m}} T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} e_{\mathbf{m}} E^i t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} E^i e_{\mathbf{m}} t^i = J_{c+r} e_{\mathbf{m}}.
\end{aligned}$$

Formulas (4.15) and (4.16) are the same as those presented in Lemma 3.3. For $\mathbf{m} \neq -\mathbf{n}$,

there is

$$\begin{aligned}
f_{\mathbf{m}} J_c &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} f_{\mathbf{m}} T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} f_{\mathbf{m}} E^i t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} (E^i f_{\mathbf{m}} - i q^{m_1 n_2} E^{i-1} g_{\mathbf{m}+\mathbf{n}} + i q^{m_2 n_1} E^{i-1} h_{\mathbf{m}+\mathbf{n}} - 2 s_{\mathbf{m}} \binom{i}{2} E^{i-2} e_{\mathbf{m}+2\mathbf{n}}) t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} E^i f_{\mathbf{m}} t^i - q^{m_1 n_2} \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!} T_{-c-r}^{[i+1]} E^i g_{\mathbf{m}+\mathbf{n}} t^{i+1} \\
&\quad + q^{m_2 n_1} \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!} T_{-c-r}^{[i+1]} E^i h_{\mathbf{m}+\mathbf{n}} t^{i+1} - s_{\mathbf{m}} \sum_{i=0}^{\infty} \frac{(-1)^{i+2}}{i!} T_{-c-r}^{[i+2]} E^i e_{\mathbf{m}+2\mathbf{n}} t^{i+2} \\
&= J_{c+r} f_{\mathbf{m}} + q^{m_1 n_2} J_{c+r} (g_{\mathbf{m}+\mathbf{n}} T_{1-c} t) - q^{m_2 n_1} J_{c+r} (h_{\mathbf{m}+\mathbf{n}} T_{1-c} t) - s_{\mathbf{m}} J_{c+r} (e_{\mathbf{m}+2\mathbf{n}} T_{1-c}^{<2>} t^2).
\end{aligned}$$

Furthermore, there is

$$\begin{aligned}
f_{-\mathbf{n}} J_c &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} f_{-\mathbf{n}} T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c+1}^{[i]} f_{-\mathbf{n}} E^i t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c+1}^{[i]} (E^i f_{-\mathbf{n}} - i q^{-n_2 n_1} E^{i-1} d - 2 q^{-n_2 n_1} \binom{i}{2} E^{i-1}) t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c+1}^{[i]} E^i f_{-\mathbf{n}} t^i - q^{-n_2 n_1} \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!} T_{-c+1}^{[i+1]} E^i d t^{i+1} \\
&\quad - q^{-n_1 n_2} \sum_{i=0}^{\infty} \frac{(-1)^{i+2}}{i!} T_{-c+1}^{[i+2]} E^{i+1} t^{i+2} \\
&= J_{c-1} f_{-\mathbf{n}} + q^{-n_2 n_1} J_{c-1} T_{1-c} d t - q^{-n_1 n_2} J_{c-1} T_{1-c}^{[2]} E t^2 \\
&= J_{c-1} f_{-\mathbf{n}} + q^{-n_1 n_2} J_{c-1} d T_{1-c} t - q^{-n_1 n_2} J_{c-1} E T_{1-c}^{<2>} t^2.
\end{aligned}$$

Thus, we obtain (4.17). Equations (4.18) to (4.20) could be obtained by the formulas (2.1), (2.3) and formulas (4.1), (4.4) and (4.5),

$$\begin{aligned}
g_{\mathbf{m}} J_c &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} g_{\mathbf{m}} T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} g_{\mathbf{m}} E^i t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} (E^i g_{\mathbf{m}} + i q^{m_2 n_1} E^{i-1} e_{\mathbf{m}+\mathbf{n}}) t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} E^i g_{\mathbf{m}} t^i + q^{m_2 n_1} \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!} T_{-c-r}^{[i+1]} E^i e_{\mathbf{m}+\mathbf{n}} t^{i+1} \\
&= J_{c+r} g_{\mathbf{m}} - q^{m_2 n_1} J_{c+r} e_{\mathbf{m}+\mathbf{n}} T_{1-c} t, \\
h_{\mathbf{m}} J_c &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} h_{\mathbf{m}} T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} h_{\mathbf{m}} E^i t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} (E^i h_{\mathbf{m}} - i q^{m_1 n_2} E^{i-1} e_{\mathbf{m}+\mathbf{n}}) t^i
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-r}^{[i]} E^i h_{\mathbf{m}} t^i - q^{m_1 n_2} \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!} T_{-c-r}^{[i+1]} E^i e_{\mathbf{m}+\mathbf{n}} t^{i+1} \\
&= J_{c+r} h_{\mathbf{m}} + q^{m_1 n_2} J_{c+r} e_{\mathbf{m}+\mathbf{n}} T_{1-c} t, \\
dJ_c &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} dT_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} dE^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} (E^i d + 2iE^i) t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} E^i dt^i + 2 \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!} T_{-c}^{[i+1]} E^{i+1} t^{i+1} = J_c d - 2J_c E T_{1-c} t.
\end{aligned}$$

□

Proof of Theorem 1.1 (2). Using equations (2.2), (2.5) and the lemmas from Lemma 4.1 to Lemma 4.3, for $\mathbf{m} \in \mathbf{Z}$, there are,

$$\begin{aligned}
\Delta(e_{\mathbf{m}}) &= \mathcal{I} \Delta_0(e_{\mathbf{m}}) \mathcal{I}^{-1} = \mathcal{I}(e_{\mathbf{m}} \otimes 1 + 1 \otimes e_{\mathbf{m}}) I = \mathcal{I} I_{-r}(e_{\mathbf{m}} \otimes 1) + \mathcal{I} I(1 \otimes e_{\mathbf{m}}) \\
&= e_{\mathbf{m}} \otimes (1 - Et)^r + 1 \otimes e_{\mathbf{m}}. \\
\Delta(g_{\mathbf{m}}) &= \mathcal{I} \Delta_0(g_{\mathbf{m}}) \mathcal{I}^{-1} = \mathcal{I}(g_{\mathbf{m}} \otimes 1 + 1 \otimes g_{\mathbf{m}}) I \\
&= \mathcal{I} I_{-r}(g_{\mathbf{m}} \otimes 1) + \mathcal{I}(I(1 \otimes g_{\mathbf{m}}) + q^{m_2 n_1} I_1(T \otimes e_{\mathbf{m}+\mathbf{n}} t)) \\
&= g_{\mathbf{m}} \otimes (1 - Et)^r + 1 \otimes g_{\mathbf{m}} + q^{m_2 n_1} T \otimes (1 - Et)^{-1} e_{\mathbf{m}+\mathbf{n}} t, \\
\Delta(h_{\mathbf{m}}) &= \mathcal{I} \Delta_0(h_{\mathbf{m}}) \mathcal{I}^{-1} = \mathcal{I}(h_{\mathbf{m}} \otimes 1 + 1 \otimes h_{\mathbf{m}}) I \\
&= \mathcal{I} I_{-r}(h_{\mathbf{m}} \otimes 1) + \mathcal{I}(I(1 \otimes h_{\mathbf{m}}) - q^{m_1 n_2} I_1(T \otimes e_{\mathbf{m}+\mathbf{n}} t)) \\
&= h_{\mathbf{m}} \otimes (1 - Et)^r + 1 \otimes h_{\mathbf{m}} - q^{m_1 n_2} T \otimes (1 - Et)^{-1} e_{\mathbf{m}+\mathbf{n}} t, \\
\Delta(d) &= \mathcal{I} \Delta_0(d) \mathcal{I}^{-1} = \mathcal{I}(d \otimes 1 + 1 \otimes d) I \\
&= \mathcal{I} I(d \otimes 1) + \mathcal{I}(I(1 \otimes d) + 2I_1(T \otimes Et)) \\
&= d \otimes 1 + 1 \otimes d + 2T \otimes (1 - Et)^{-1} Et, \\
\Delta(d_1) &= \mathcal{I} \Delta_0(d_1) \mathcal{I}^{-1} = \mathcal{I}(d_1 \otimes 1 + 1 \otimes d_1) I \\
&= \mathcal{I} I(d_1 \otimes 1) + \mathcal{I}(n_1 I_1(T \otimes Et) + I(1 \otimes d_1)) \\
&= d_1 \otimes 1 + 1 \otimes d_1 - n_1 T \otimes 1 + n_1 T \otimes (1 - Et)^{-1}, \\
\Delta(d_2) &= \mathcal{I} \Delta_0(d_2) \mathcal{I}^{-1} = \mathcal{I}(d_2 \otimes 1 + 1 \otimes d_2) I \\
&= \mathcal{I} I(d_2 \otimes 1) + \mathcal{I}(n_2 I_1(T \otimes Et) + I(1 \otimes d_2)) \\
&= d_2 \otimes 1 + 1 \otimes d_2 - n_2 T \otimes 1 + n_2 T \otimes (1 - Et)^{-1}, \\
S(e_{\mathbf{m}}) &= \mathcal{J} s_0(e_{\mathbf{m}}) J = -\mathcal{J} e_{\mathbf{m}} J = -\mathcal{J} J_r(e_{\mathbf{m}}) = -(1 - Et)^r e_{\mathbf{m}}, \\
S(g_{\mathbf{m}}) &= \mathcal{J} s_0(g_{\mathbf{m}}) J = -\mathcal{J}(g_{\mathbf{m}}) J = -\mathcal{J}(J_r g_{\mathbf{m}} - q^{m_2 n_1} J_r(e_{\mathbf{m}+\mathbf{n}} T_1 t)) \\
&= -(1 - Et)^r g_{\mathbf{m}} + q^{m_2 n_1} (1 - Et)^r e_{\mathbf{m}+\mathbf{n}} T_1 t, \\
S(h_{\mathbf{m}}) &= \mathcal{J} s_0(h_{\mathbf{m}}) J = -\mathcal{J}(h_{\mathbf{m}}) J = -\mathcal{J}(J_r h_{\mathbf{m}} + q^{m_1 n_2} J_r(e_{\mathbf{m}+\mathbf{n}} T_1 t)) \\
&= -(1 - Et)^r h_{\mathbf{m}} - q^{m_1 n_2} (1 - Et)^r e_{\mathbf{m}+\mathbf{n}} T_1 t,
\end{aligned}$$

$$\begin{aligned}
S(d) &= \mathcal{J}s_0(d)J = -\mathcal{J}dJ = -\mathcal{J}(Jd - 2J(ET_1t)) = -d + 2ET_1t, \\
S(d_1) &= \mathcal{J}s_0(d_1)J = -\mathcal{J}d_1J = -\mathcal{J}(Jd_1 - n_1JTEt) = -d_1 + n_1TEt, \\
S(d_2) &= \mathcal{J}s_0(d_2)J = -\mathcal{J}d_2J = -\mathcal{J}(Jd_2 - n_2JTEt) = -d_2 + n_2TEt.
\end{aligned}$$

Moreover, the following equations are also necessary to the theorem, where $\mathbf{m} + \mathbf{n} \neq 0$,

$$\begin{aligned}
\Delta(f_{\mathbf{m}}) &= \mathcal{I}\Delta_0(f_{\mathbf{m}})\mathcal{I}^{-1} = \mathcal{I}(f_{\mathbf{m}} \otimes 1 + 1 \otimes f_{\mathbf{m}})I \\
&= \mathcal{I}I_{-r}(f_{\mathbf{m}} \otimes 1) + \mathcal{I}I(1 \otimes f_{\mathbf{m}}) - q^{m_1n_2}\mathcal{I}I_1(T \otimes g_{\mathbf{m}+\mathbf{n}}t) \\
&\quad + q^{m_2n_1}\mathcal{I}I_1(T \otimes h_{\mathbf{m}+\mathbf{n}}t) - s_{\mathbf{m}}\mathcal{I}I_2(T^{<2>} \otimes e_{\mathbf{m}+2\mathbf{n}}t^2) \\
&= f_{\mathbf{m}} \otimes (1 - Et)^r + 1 \otimes f_{\mathbf{m}} - q^{m_1n_2}T \otimes (1 - Et)^{-1}g_{\mathbf{m}+\mathbf{n}}t \\
&\quad + q^{m_2n_1}T \otimes (1 - Et)^{-1}h_{\mathbf{m}+\mathbf{n}}t - s_{\mathbf{m}}T^{<2>} \otimes (1 - Et)^{-2}e_{\mathbf{m}+2\mathbf{n}}t^2, \\
\Delta(f_{-\mathbf{n}}) &= \mathcal{I}\Delta_0(f_{-\mathbf{n}})\mathcal{I}^{-1} = \mathcal{I}(f_{-\mathbf{n}} \otimes 1 + 1 \otimes f_{-\mathbf{n}})I \\
&= \mathcal{I}I_1(f_{-\mathbf{n}} \otimes 1) + \mathcal{I}(I(1 \otimes f_{-\mathbf{n}}) - q^{-n_2n_1}I_1(T \otimes dt) - q^{-n_1n_2}I_2(T^{<2>} \otimes Et^2)) \\
&= f_{-\mathbf{n}} \otimes (1 - Et)^{-1} + 1 \otimes f_{-\mathbf{n}} - q^{-n_2n_1}T \otimes (1 - Et)^{-1}dt \\
&\quad - q^{-n_1n_2}T^{<2>} \otimes (1 - Et)^{-2}Et^2, \\
S(f_{\mathbf{m}}) &= \mathcal{J}s_0(f_{\mathbf{m}})J = -\mathcal{J}f_{\mathbf{m}}J \\
&= \mathcal{J}J_r(q^{m_2n_1}h_{\mathbf{m}+\mathbf{n}}T_1t - q^{m_1n_2}g_{\mathbf{m}+\mathbf{n}}T_1t - f_{\mathbf{m}} + s_{\mathbf{m}}e_{\mathbf{m}+2\mathbf{n}}T_1^{<2>}t^2) \\
&= q^{m_2n_1}(1 - Et)^r h_{\mathbf{m}+\mathbf{n}}T_1t - q^{m_1n_2}(1 - Et)^r g_{\mathbf{m}+\mathbf{n}}T_1t \\
&\quad - (1 - Et)^r f_{\mathbf{m}} + s_{\mathbf{m}}(1 - Et)^r e_{\mathbf{m}+2\mathbf{n}}T_1^{<2>}t^2, \\
S(f_{-\mathbf{n}}) &= \mathcal{J}s_0(f_{-\mathbf{n}})J = -\mathcal{J}f_{-\mathbf{n}}J \\
&= -\mathcal{J}(J_{-1}f_{-\mathbf{n}} + q^{-n_1n_2}J_{-1}(dT_1t) - q^{-n_1n_2}J_{-1}ET_1^{<2>}t^2) \\
&= -(1 - Et)^{-1}f_{-\mathbf{n}} - q^{-n_1n_2}(1 - Et)^{-1}dT_1t + q^{-n_1n_2}(1 - Et)^{-1}ET_1^{<2>}t^2.
\end{aligned}$$

□

5. Proof of Theorem 1.1 (3)

In this section, we take $T = \frac{1}{2}d$ and $E = e_{\mathbf{n}}$ for $\mathbf{n} = (n_1, n_2) \in \mathbf{Z}$ such that $[T, E] = E$. The expressions only referring to E in Section 4 are also tenable in this section, such as expressions from (4.7) to (4.13) in Lemma 4.2. For $\mathbf{m} = (m_1, m_2) \in \mathbf{Z}$, $r = x_1m_1 + x_2m_2$, denote

$$s_{\mathbf{m}} = q^{n_2m_1 + n_1m_2 + n_1n_2}.$$

Lemma 5.1. *The following identities hold in $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})$:*

$$e_{\mathbf{m}}T_c^{[i]} = T_{c-1}^{[i]}e_{\mathbf{m}}, \quad e_{\mathbf{m}}T_c^{<i>} = T_{c-1}^{<i>}e_{\mathbf{m}}, \quad (5.1)$$

$$f_{\mathbf{m}}T_c^{[i]} = T_{c+1}^{[i]}f_{\mathbf{m}}, \quad f_{\mathbf{m}}T_c^{<i>} = T_{c+1}^{<i>}f_{\mathbf{m}}, \quad (5.2)$$

$$g_{\mathbf{m}}T_c^{[i]} = T_c^{[i]}g_{\mathbf{m}}, \quad g_{\mathbf{m}}T_c^{<i>} = T_c^{<i>}g_{\mathbf{m}}, \quad (5.3)$$

$$h_{\mathbf{m}}T_c^{[i]} = T_c^{[i]}h_{\mathbf{m}}, \quad h_{\mathbf{m}}T_c^{<i>} = T_c^{<i>}h_{\mathbf{m}}, \quad (5.4)$$

$$dT_c^{[i]} = T_c^{[i]}d, \quad dT_c^{<i>} = T_c^{<i>}d, \quad (5.5)$$

$$d_1T_c^{[i]} = T_c^{[i]}d_1, \quad d_1T_c^{<i>} = T_c^{<i>}d_1, \quad (5.6)$$

$$d_2T_c^{[i]} = T_c^{[i]}d_2, \quad d_2T_c^{<i>} = T_c^{<i>}d_2. \quad (5.7)$$

Proof. Since $[T, e_{\mathbf{m}}] = e_{\mathbf{m}}$, $[T, f_{\mathbf{m}}] = -f_{\mathbf{m}}$, $[T, g_{\mathbf{m}}] = [T, h_{\mathbf{m}}] = [T, d] = [T, d_1] = [T, d_2] = 0$, we obtain equations from (5.1) to (5.7). \square

Lemma 5.2. *The following identities hold in $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})$:*

$$(e_{\mathbf{m}} \otimes 1)I_c = I_{c-1}(e_{\mathbf{m}} \otimes 1), \quad (5.8)$$

$$(f_{\mathbf{m}} \otimes 1)I_c = I_{c+1}(f_{\mathbf{m}} \otimes 1), \quad (5.9)$$

$$(g_{\mathbf{m}} \otimes 1)I_c = I_c(g_{\mathbf{m}} \otimes 1), \quad (5.10)$$

$$(h_{\mathbf{m}} \otimes 1)I_c = I_c(h_{\mathbf{m}} \otimes 1), \quad (5.11)$$

$$(d \otimes 1)I_c = I_c(d \otimes 1), \quad (5.12)$$

$$(d_1 \otimes 1)I_c = I_c(d_1 \otimes 1), \quad (5.13)$$

$$(d_2 \otimes 1)I_c = I_c(d_2 \otimes 1). \quad (5.14)$$

Proof. The equation (5.8) can be obtained as following by equation (5.1),

$$(e_{\mathbf{m}} \otimes 1)I_c = \sum_{i=0}^{\infty} \frac{1}{i!} e_{\mathbf{m}} T_c^{<i>} \otimes E^i t^i = \sum_{i=0}^{\infty} \frac{1}{i!} T_{c-1}^{<i>} e_{\mathbf{m}} \otimes E^i t^i = I_{c-1}(e_{\mathbf{m}} \otimes 1).$$

By the similar method, we obtain equations from (5.9) to (5.14). \square

Lemma 5.3. *The following identities hold in $\mathcal{U}(\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)})$:*

$$e_{\mathbf{m}}J_c = J_{c+1}e_{\mathbf{m}}, \quad (5.15)$$

$$d_1J_c = J_cd_1 - n_1J_cT_{-c}Et, \quad (5.16)$$

$$d_2J_c = J_cd_2 - n_2J_cT_{-c}Et, \quad (5.17)$$

$$f_{\mathbf{m}}J_c = \begin{cases} q^{m_1n_2}J_{c-1}(T_{-c+1}g_{\mathbf{m}+\mathbf{n}}t) - q^{m_2n_1}J_{c-1}(h_{\mathbf{m}+\mathbf{n}}T_{1-c}t) \\ + J_{c-1}f_{\mathbf{m}} - s_{\mathbf{m}}J_{c-1}(e_{\mathbf{m}+2\mathbf{n}}T_{1-c}^{<2>}t^2), & \mathbf{m} + \mathbf{n} \neq 0, \\ J_{c-1}f_{-\mathbf{n}} + q^{-n_1n_2}J_{c-1}(dT_{1-c}t) - q^{-n_1n_2}J_{c-1}ET_{1-c}^{<2>}t^2, & \mathbf{m} + \mathbf{n} = 0, \end{cases} \quad (5.18)$$

$$g_{\mathbf{m}}J_c = J_cg_{\mathbf{m}} - q^{n_1m_2}J_c(e_{\mathbf{m}+\mathbf{n}}T_{1-c}t), \quad (5.19)$$

$$h_{\mathbf{m}}J_c = J_ch_{\mathbf{m}} + q^{n_2m_1}J_c(e_{\mathbf{m}+\mathbf{n}}T_{1-c}t), \quad (5.20)$$

$$dJ_c = J_cd - 2J_c(ET_{1-c}t). \quad (5.21)$$

Proof. Using the formula (2.3), (4.1) and (5.1), we have

$$e_{\mathbf{m}}J_c = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} e_{\mathbf{m}} T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-1}^{[i]} e_{\mathbf{m}} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c-1}^{[i]} E^i e_{\mathbf{m}} t^i = J_{c+1} e_{\mathbf{m}}.$$

Formulas (5.16) and (5.17) are the same as those presented in Lemma 3.3. For $\mathbf{m} \neq -\mathbf{n}$, there is

$$\begin{aligned} f_{\mathbf{m}}J_c &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} f_{\mathbf{m}} T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c+1}^{[i]} f_{\mathbf{m}} E^i t^i \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c+1}^{[i]} (E^i f_{\mathbf{m}} - iq^{m_1n_2} E^{i-1} g_{\mathbf{m}+\mathbf{n}} + iq^{m_2n_1} E^{i-1} h_{\mathbf{m}+\mathbf{n}} - 2s_{\mathbf{m}} \binom{i}{2} E^{i-2} e_{\mathbf{m}+2\mathbf{n}}) t^i \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c+1}^{[i]} E^i f_{\mathbf{m}} t^i - q^{m_1n_2} \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!} T_{-c+1}^{[i+1]} E^i g_{\mathbf{m}+\mathbf{n}} t^{i+1} \\ &\quad + q^{m_2n_1} \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!} T_{-c+1}^{[i+1]} E^i h_{\mathbf{m}+\mathbf{n}} t^{i+1} - s_{\mathbf{m}} \sum_{i=0}^{\infty} \frac{(-1)^{i+2}}{i!} T_{-c+1}^{[i+2]} E^i e_{\mathbf{m}+2\mathbf{n}} t^{i+2} \\ &= J_{c-1} f_{\mathbf{m}} + q^{m_1n_2} J_{c-1} (g_{\mathbf{m}+\mathbf{n}} T_{1-c} t) - q^{m_2n_1} J_{c-1} (h_{\mathbf{m}+\mathbf{n}} T_{1-c} t) - s_{\mathbf{m}} J_{c-1} (e_{\mathbf{m}+2\mathbf{n}} T_{1-c}^{<2>} t^2). \end{aligned}$$

Furthermore, there is

$$\begin{aligned} f_{-\mathbf{n}}J_c &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} f_{-\mathbf{n}} T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c+1}^{[i]} f_{-\mathbf{n}} E^i t^i \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c+1}^{[i]} (E^i f_{-\mathbf{n}} - iq^{-n_2n_1} E^{i-1} d - 2q^{-n_2n_1} \binom{i}{2} E^{i-1}) t^i \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c+1}^{[i]} E^i f_{-\mathbf{n}} t^i - q^{-n_2n_1} \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!} T_{-c+1}^{[i+1]} E^i d t^{i+1} \\ &\quad - q^{-n_1n_2} \sum_{i=0}^{\infty} \frac{(-1)^{i+2}}{i!} T_{-c+1}^{[i+2]} E^{i+1} t^{i+2} \\ &= J_{c-1} f_{-\mathbf{n}} + q^{-n_2n_1} J_{c-1} T_{1-c} d t - q^{-n_1n_2} J_{c-1} T_{1-c}^{[2]} E t^2 \\ &= J_{c-1} f_{-\mathbf{n}} + q^{-n_1n_2} J_{c-1} d T_{1-c} t - q^{-n_1n_2} J_{c-1} E T_{1-c}^{<2>} t^2. \end{aligned}$$

Thus, we obtain (5.18). Equations (5.19) to (5.21) could be obtained by the formulas (2.1), (2.3) and formulas from (5.3) to (5.5),

$$g_{\mathbf{m}}J_c = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} g_{\mathbf{m}} T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} g_{\mathbf{m}} E^i t^i$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} (E^i g_{\mathbf{m}} + i q^{m_2 n_1} E^{i-1} e_{\mathbf{m}+\mathbf{n}}) t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} E^i g_{\mathbf{m}} t^i + q^{m_2 n_1} \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!} T_{-c}^{[i+1]} E^i e_{\mathbf{m}+\mathbf{n}} t^{i+1} \\
&= J_c g_{\mathbf{m}} - q^{m_2 n_1} J_c e_{\mathbf{m}+\mathbf{n}} T_{1-c} t, \\
h_{\mathbf{m}} J_c &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} h_{\mathbf{m}} T_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} h_{\mathbf{m}} E^i t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} (E^i h_{\mathbf{m}} - i q^{m_1 n_2} E^{i-1} e_{\mathbf{m}+\mathbf{n}}) t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} E^i h_{\mathbf{m}} t^i - q^{m_1 n_2} \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!} T_{-c}^{[i+1]} E^i e_{\mathbf{m}+\mathbf{n}} t^{i+1} \\
&= J_c h_{\mathbf{m}} + q^{m_1 n_2} J_c e_{\mathbf{m}+\mathbf{n}} T_{1-c} t, \\
dJ_c &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} dT_{-c}^{[i]} E^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} dE^i t^i = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} (E^i d + 2i E^i) t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-c}^{[i]} E^i d t^i + 2 \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!} T_{-c}^{[i+1]} E^{i+1} t^{i+1} = J_c d - 2J_c E T_{1-c} t.
\end{aligned}$$

□

Proof of Theorem 1.1 (3). Using equations (2.2), (2.5) and all the lemmas above on this section, we obtain, for $\mathbf{m} + \mathbf{n} \neq 0$,

$$\begin{aligned}
\Delta(f_{\mathbf{m}}) &= \mathcal{I} \Delta_0(f_{\mathbf{m}}) \mathcal{I}^{-1} = \mathcal{I} (f_{\mathbf{m}} \otimes 1 + 1 \otimes f_{\mathbf{m}}) I \\
&= \mathcal{I} I_1(f_{\mathbf{m}} \otimes 1) + \mathcal{I} I(1 \otimes f_{\mathbf{m}}) - q^{m_1 n_2} \mathcal{I} I_1(T \otimes g_{\mathbf{m}+\mathbf{n}} t) \\
&\quad + q^{m_2 n_1} \mathcal{I} I_1(T \otimes h_{\mathbf{m}+\mathbf{n}} t) - s_{\mathbf{m}} \mathcal{I} I_2(T^{<2>} \otimes e_{\mathbf{m}+2\mathbf{n}} t^2) \\
&= f_{\mathbf{m}} \otimes (1 - Et)^{-1} + 1 \otimes f_{\mathbf{m}} - q^{m_1 n_2} T \otimes (1 - Et)^{-1} g_{\mathbf{m}+\mathbf{n}} t \\
&\quad + q^{m_2 n_1} T \otimes (1 - Et)^{-1} h_{\mathbf{m}+\mathbf{n}} t - s_{\mathbf{m}} T^{<2>} \otimes (1 - Et)^{-2} e_{\mathbf{m}+2\mathbf{n}} t^2, \\
\Delta(f_{-\mathbf{n}}) &= \mathcal{I} \Delta_0(f_{-\mathbf{n}}) \mathcal{I}^{-1} = \mathcal{I} (f_{-\mathbf{n}} \otimes 1 + 1 \otimes f_{-\mathbf{n}}) I \\
&= \mathcal{I} I_1(f_{-\mathbf{n}} \otimes 1) - q^{-n_2 n_1} \mathcal{I} I_1(T \otimes dt) \\
&\quad + \mathcal{I} I(1 \otimes f_{-\mathbf{n}}) - q^{-n_1 n_2} \mathcal{I} I_2(T^{<2>} \otimes Et^2) \\
&= f_{-\mathbf{n}} \otimes (1 - Et)^{-1} - q^{-n_2 n_1} T \otimes (1 - Et)^{-1} dt \\
&\quad + 1 \otimes f_{-\mathbf{n}} - q^{-n_1 n_2} T^{<2>} \otimes (1 - Et)^{-2} t^2, \\
S(f_{\mathbf{m}}) &= \mathcal{J} s_0(f_{\mathbf{m}}) J = -\mathcal{J} f_{\mathbf{m}} J \\
&= q^{m_2 n_1} \mathcal{J} J_{-1}(h_{\mathbf{m}+\mathbf{n}} T_1 t) - q^{m_1 n_2} \mathcal{J} J_{-1}(T_1 g_{\mathbf{m}+\mathbf{n}} t) \\
&\quad - \mathcal{J} J_{-1} f_{\mathbf{m}} + s_{\mathbf{m}} \mathcal{J} J_{-1}(e_{\mathbf{m}+2\mathbf{n}} T_1^{<2>} t^2) \\
&= q^{m_2 n_1} (1 - Et)^{-1} h_{\mathbf{m}+\mathbf{n}} T_1 t - q^{m_1 n_2} (1 - Et)^{-1} T_1 g_{\mathbf{m}+\mathbf{n}} t
\end{aligned}$$

$$\begin{aligned}
& -(1 - Et)^{-1}f_{\mathbf{m}} + s_{\mathbf{m}}(1 - Et)^{-1}e_{\mathbf{m}+2\mathbf{n}}T_1^{<2>}t^2, \\
S(f_{-\mathbf{n}}) &= \mathcal{J}s_0(f_{-\mathbf{n}})J = -\mathcal{J}f_{-\mathbf{n}}J \\
&= -\mathcal{J}(J_{-1}f_{-\mathbf{n}} + q^{-n_1n_2}J_{-1}(dT_1t) - q^{-n_1n_2}J_{-1}ET_1^{<2>}t^2) \\
&= -(1 - Et)^{-1}f_{-\mathbf{n}} - q^{-n_1n_2}(1 - Et)^{-1}dT_1t + q^{-n_1n_2}(1 - Et)^{-1}ET_1^{<2>}t^2.
\end{aligned}$$

What is more, for any $\mathbf{m} \in \mathbf{Z}$, we also obtain,

$$\begin{aligned}
\Delta(e_{\mathbf{m}}) &= \mathcal{I}\Delta_0(e_{\mathbf{m}})\mathcal{I}^{-1} = \mathcal{I}(e_{\mathbf{m}} \otimes 1 + 1 \otimes e_{\mathbf{m}})I \\
&= \mathcal{I}I_{-1}(e_{\mathbf{m}} \otimes 1) + \mathcal{I}I(1 \otimes e_{\mathbf{m}}) \\
&= e_{\mathbf{m}} \otimes (1 - Et) + 1 \otimes e_{\mathbf{m}}, \\
\Delta(g_{\mathbf{m}}) &= \mathcal{I}\Delta_0(g_{\mathbf{m}})\mathcal{I}^{-1} = \mathcal{I}(g_{\mathbf{m}} \otimes 1 + 1 \otimes g_{\mathbf{m}})I \\
&= \mathcal{I}I(g_{\mathbf{m}} \otimes 1) + \mathcal{I}(I(1 \otimes g_{\mathbf{m}}) + q^{m_2n_1}I_1(T \otimes e_{\mathbf{m}+\mathbf{n}}t)) \\
&= g_{\mathbf{m}} \otimes 1 + 1 \otimes g_{\mathbf{m}} + q^{m_2n_1}T \otimes (1 - Et)^{-1}e_{\mathbf{m}+\mathbf{n}}t, \\
\Delta(h_{\mathbf{m}}) &= \mathcal{I}\Delta_0(h_{\mathbf{m}})\mathcal{I}^{-1} = \mathcal{I}(h_{\mathbf{m}} \otimes 1 + 1 \otimes h_{\mathbf{m}})I \\
&= \mathcal{I}I_0(h_{\mathbf{m}} \otimes 1) + \mathcal{I}(I(1 \otimes h_{\mathbf{m}}) - q^{m_1n_2}I_1(T \otimes e_{\mathbf{m}+\mathbf{n}}t)) \\
&= h_{\mathbf{m}} \otimes 1 + 1 \otimes h_{\mathbf{m}} - q^{m_1n_2}T \otimes (1 - Et)^{-1}e_{\mathbf{m}+\mathbf{n}}t, \\
\Delta(d) &= \mathcal{I}\Delta_0(d)\mathcal{I}^{-1} = \mathcal{I}(d \otimes 1 + 1 \otimes d)I \\
&= \mathcal{I}I(d \otimes 1) + \mathcal{I}(I(1 \otimes d) + 2I_1(T \otimes Et)) \\
&= d \otimes 1 + 1 \otimes d + 2T \otimes (1 - Et)^{-1}Et, \\
\Delta(d_1) &= \mathcal{I}\Delta_0(d_1)\mathcal{I}^{-1} = \mathcal{I}(d_1 \otimes 1 + 1 \otimes d_1)I \\
&= \mathcal{I}I(d_1 \otimes 1) + \mathcal{I}(n_1I_1(T \otimes Et) + I(1 \otimes d_1)) \\
&= d_1 \otimes 1 + 1 \otimes d_1 + n_1T \otimes (1 - Et)^{-1} - n_1T \otimes 1, \\
\Delta(d_2) &= \mathcal{I}\Delta_0(d_2)\mathcal{I}^{-1} = \mathcal{I}(d_2 \otimes 1 + 1 \otimes d_2)I \\
&= \mathcal{I}I(d_2 \otimes 1) + \mathcal{I}(n_2I_1(T \otimes Et) + I(1 \otimes d_2)) \\
&= d_2 \otimes 1 + 1 \otimes d_2 + n_2T \otimes (1 - Et)^{-1} - n_2T \otimes 1. \\
S(e_{\mathbf{m}}) &= \mathcal{J}s_0(e_{\mathbf{m}})J = -\mathcal{J}e_{\mathbf{m}}J = -\mathcal{J}J_1e_{\mathbf{m}} = -(1 - Et)e_{\mathbf{m}}, \\
S(g_{\mathbf{m}}) &= \mathcal{J}s_0(g_{\mathbf{m}})J = -\mathcal{J}g_{\mathbf{m}}J = -\mathcal{J}(Jg_{\mathbf{m}} - q^{m_2n_1}J(e_{\mathbf{m}+\mathbf{n}}T_1t)) \\
&= -g_{\mathbf{m}} + q^{m_2n_1}e_{\mathbf{m}+\mathbf{n}}T_1t, \\
S(h_{\mathbf{m}}) &= \mathcal{J}s_0(h_{\mathbf{m}})J = -\mathcal{J}h_{\mathbf{m}}J = -\mathcal{J}(Jh_{\mathbf{m}} + q^{n_2m_1}J(e_{\mathbf{m}+\mathbf{n}}T_1t)) \\
&= -h_{\mathbf{m}} - q^{n_2m_1}e_{\mathbf{m}+\mathbf{n}}T_1t, \\
S(d) &= \mathcal{J}s_0(d)J = -\mathcal{J}dJ = -\mathcal{J}(Jd - 2J(ET_1t)) = -d + 2ET_1t, \\
S(d_1) &= \mathcal{J}s_0(d_1)J = -\mathcal{J}d_1J = -\mathcal{J}(Jd_1 - n_1JT_1Et) = -d_1 + n_1T_1Et, \\
S(d_2) &= \mathcal{J}s_0(d_2)J = -\mathcal{J}d_2J = -\mathcal{J}(Jd_2 - n_2JT_1Et) = -d_2 + n_2T_1Et.
\end{aligned}$$

□

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References

- [1] B. Allison, Y. Gao, The root system and the core of an extended affine Lie algebra, *Selecta Math. (N.S.)*, **7**(2) (2001), 149–212.
- [2] B. Allison, S. Azam, S. Berman, Y. Gao, A. Pianzola, Extended affine Lie algebras and their root systems, *Mem. Amer. Math. Soc.*, **126**(603) (1997).
- [3] Y. Billig, Representations of toroidal extended affine Lie algebras, *J. Algebra*, **308** (2007), 252–269.
- [4] S. Berman, Y. Gao, Y. Krylyuk, Quantum tori and the structure of elliptic quasi-simple Lie algebras, *J. Funct. Anal.*, **135** (1996) 339–389.
- [5] S. Berman, Y. Gao, Y. Krylyuk, E. Neher, The alternative torus and the structure of elliptic quasisimple Lie algebras of type A_2 , *Trans. Amer. Math. Soc.*, **347**(11) (1995), 4315–4363.
- [6] Y. Billig, M. Lau, Irreducible modules for extended affine Lie algebras, *J. Algebra*, **327** (2011) 208–235.
- [7] Y. Cheng, Y. Su, Quantization of Lie algebras of block type, *Acta Mathematica Scientia*, **30**(4), (2010), 1134–1142.
- [8] V. Drinfel’d, Constant quasiclassical solutions of the Yang-Baxter quantum equation, *Soviet Math. Dokl.*, **28**(3) (1983), 667–671.
- [9] V. Drinfel’d, Quantum groups, Proceedings ICM (Berkeley 1986), Providence: Amer Math Soc, (1987) 789–820.
- [10] V. Drinfel’d, Quantum groups, in: *Proceeding of the International Congress of Mathematicians*, Vol. 1, 2, Berkeley, Calif. 1986, Amer. Math. Soc., Providence, RI, 1987, 798–820.
- [11] P. Etingof, D. Kazhdan, Quantization of Lie bialgebras I, *Selecta Math.*, **2**(1) (1996) 1–41.
- [12] P. Etingof, D. Kazhdan, Quantization of Lie bialgebras II, *Selecta Math.*, **2**(4) (1998) 213–231.
- [13] P. Etingof, O. Schiffmann, Lectures on Quantum Groups, 2ed, International Press, USA, 2002.
- [14] Y. Gao, Representations of extended affine Lie algebras coordinatized by certain quantum tori, *Compositio Math.*, **123**(1) (2000), 1–25.
- [15] Y. Gao, Fermionic and bosonic representations of the extended affine Lie algebra $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$, *Cananian Math. Bull.*, **45** (2002), 623–633.
- [16] Y. Gao, Z. Zeng, Hermitian representations of the extended affine Lie algebra $\widetilde{\mathfrak{gl}_2(\mathbb{C}_q)}$, *Adv. Math.*, **207**(1) (2006), 244–265.
- [17] C. Grunspan, Quantizations of the Witt algebra and of simple Lie algebras in characteristic p , *J. Algebra*, **280** (2004), 145–161.
- [18] A. Giaquinto, J. Zhang, Bialgebra action, twists and universal deformation formulas, *J. Pure Appl. Algebra*, **128**(2) (1998), 133–151.

- [19] R. Høegh-Krohn, B. Torresani, Classification and construction of quasi-simple Lie algebras, *J. Funct. Anal.*, **89** (1990), 106–136.
- [20] C. Jiang, D. Meng, The derivation algebra of the associative algebra $C_q[X, Y, X^{-1}, Y^{-1}]$, *Comm. Algebra*, **26**(6) (1998), 1723–1736.
- [21] J. Li, Y. Su, Quantizations of the W-algebra $W(2, 2)$, *Acta Mathematica Sinica*, **27**(4) (2011), 647–656.
- [22] W. Lin, Y. Su, Modules for the core of extended affine Lie algebras of type A_1 with coordinates in rank 2 quantum tori, *Pacific J. Math.*, **242**(1) (2009), 143–166.
- [23] E. Neher, Extended affine Lie algebras, *C. R. Math. Acad. Sci. Soc. R. Can.*, **26** (2004) 90–96.
- [24] G. Song, Y. Su, Y. Wu, Quantization of generalized Virasoro-like algebras, *Linear Algebra and its Applications*, **428**, (2008), 2888–2899.
- [25] Y. Xu, J. Li, Lie bialgebra structures on the extended affine Lie algebra $\widetilde{\mathfrak{sl}_2(\mathbb{C}_q)}$, arXiv:1102.5226v2.